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Derivation of Green's function of a spin Calogero–Sutherland model by Uglov's method

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Abstract

The hole propagator of a spin 1/2 Calogero–Sutherland model is derived using Uglov's method, which maps the exact eigenfunctions of the model, called the Yangian Gelfand-Zetlin basis, to a limit of Macdonald polynomials (gl_2-Jack polynomials). To apply this mapping method to the calculation of 1-particle Green's function, we confirm that the sum of the field annihilation operator $\psi_\uparrow + \psi_\downarrow$ on a Yangian Gelfand-Zetlin basis is transformed to the field annihilation operator ψ on gl_2-Jack polynomials by the mapping. The resultant expression for the hole propagator for a finite-size system is written in terms of renormalized momenta and spin of quasi-holes, and the expression in the thermodynamic limit coincides with the earlier result derived by another method. We also discuss the singularity of the spectral function for a specific coupling parameter where the hole propagator of the spin Calogero–Sutherland model becomes equivalent to the dynamical colour correlation function of an $SU(3)$ Haldane–Shastry model.

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1. Introduction

The Calogero–Sutherland model [1–9] is a one-dimensional quantum system of particles with two-body interaction inversely proportional to the square of distance. From a theoretical point of view, the Calogero–Sutherland model has attracted extensive attention in relation to fractional exclusion statistics [10–12], Jack polynomials [13, 14], Tomonaga–Luttinger liquid [15], collective field theory [16, 17] and matrix models [18]. An intriguing property of the Calogero–Sutherland model lies in the fact that the exact expressions of two-point dynamical correlation functions have been obtained for a whole range of time and space [18–27]. The expressions for dynamical correlation functions are much simpler than those of Bethe solvable models [28]. Through the exact explicit expressions of the spectral functions of the Calogero–Sutherland model, dynamical properties of one-dimensional quantum systems have been discussed [29, 30].

The dynamics of the Calogero–Sutherland model for an arbitrary rational interaction parameter was studied about a decade ago using the mathematical properties of symmetric Jack polynomials. A density correlation function, a hole propagator and a particle propagator have been calculated combinatorically using partition and a Young diagram, for a finite system, and also in a thermodynamic limit [22–25, 27].

In this paper, we consider a spin generalization of the Calogero–Sutherland model [31–33], which is the model of particles with spin $1/2$ as internal degrees of freedom. In the following, we call this model the 'spin Calogero–Sutherland model' while the original model is called the 'scalar Calogero–Sutherland model'. Spin Calogero–Sutherland models are even more important than the scalar model in the sense that (i) the spin Calogero–Sutherland models have Yangian symmetry [34, 35] as an internal symmetry, (ii) it reduces to the Haldane–Shastry model [36, 37] in the limit of an infinite coupling parameter [38, 39] and (iii) it realizes the Tomonaga–Luttinger liquid (of particles with spin internal symmetry) in the systems with a finite number of particles in the simplest manner [32, 40].

Exact results of dynamical correlation functions such as a hole propagator [41, 42], density correlation [43, 44] and spin correlation function [43–46] have been obtained in a spin Calogero–Sutherland model. However, the particle propagator of this model has not been derived. With knowledge of hole and particle propagators, we can arrive at full understanding of the dynamics and elementary excitations. From the hole and particle propagators, furthermore, we can construct a 1-particle causal Green's function, which provides a nontrivial starting point of many-body perturbation theory. The derivation of exact results applicable both to hole and particle propagators, therefore, is highly required from the above point of view. We address this issue in the present paper.

There are two types of the expression of the wavefunctions of the spin Calogero–Sutherland model. The first one is given by a Jack polynomial with prescribed symmetry [47, 48]. This polynomial is constructed by partially symmetrization or anti-symmetrization of non-symmetric Jack polynomials [49, 50], which are the simultaneous eigenfunctions of the integrals of motion of this model, that is Cherednik–Dunkl operators [51, 52]. Jack polynomials with prescribed symmetry form an orthogonal basis of the Hilbert space with a specific spin configuration. The second one is related to the Yangian symmetry [34, 35] of the spin Calogero–Sutherland model. An orthogonal basis of the Fock space including spin degrees of freedom with a fixed particle number is called the Yangian Gelfand-Zetlin basis [53, 54]

Dynamical correlation functions of the spin Calogero–Sutherland model were calculated in two methods according to the two types of eigenfunctions. One way uses Jack polynomials with prescribed symmetry as eigenfunctions, and the dynamical correlation functions are calculated by using the relations derived from the non-symmetric Jack polynomials. With this method, the hole propagator [41, 42] has been derived by one of the authors and his collaborator. Recently [55], it was found that the dynamical density correlation of the spin Calogero–Sutherland model can be derived with Jack polynomials with prescribed symmetry and the method used in [56]. However, no formulae necessary to the particle propagator have been obtained in the theory of Jack polynomials with prescribed symmetry.

The other way uses the Yangian Gelfand-Zetlin basis as an eigenfunction together with the mapping from the Yangian Gelfand-Zetlin basis to symmetric polynomials [43]. The relations of this polynomial necessary to calculate the dynamical correlation function are derived from those of Macdonald polynomials [14]. The density correlation function and the spin correlation function have been derived with this method [43, 44]. We refer to the latter method as Uglov's method. For a decade, it has been an unresolved issue whether Uglov's method is applicable to the calculation of the 1-particle Green's function. In Uglov's method, wavefunctions of multi-component particles are mapped to those of single-component particles. It is not obvious that how the field operator in the 1-particle Green's function is transformed under this mapping, in contrast to the density operator or spin operator considered in [43].

The main purpose of this paper is to show that Uglov's method is applicable to the single-particle Green's function. More explicitly, we give the transformation of the field operator under Uglov's mapping from a multi-component system to a single-component one. As an application, the hole propagator is calculated by using this method. With introducing renormalized momenta and spin variables of quasi-holes, the expression for a hole propagator in a finite-sized system becomes much simpler than that derived in [42]. We confirm that the expression in the thermodynamic limit recovers the earlier result [42]. Furthermore, we discuss the spectral function of a hole propagator for a specific coupling constant ($\lambda = 1$ in the notations, we will introduce in the following chapters). At this coupling parameter, the hole propagator of a spin Calogero–Sutherland model is equivalent to the dynamical colour correlation function [45, 46] of an $SU(3)$ Haldane–Shastry model [31, 32, 57] as shown by Arikawa [58]. In the same way, as the hole propagator, a particle propagator can be mapped to that of the single-component model. However, it is more involved to take the thermodynamic limit of a particle propagator than a hole propagator. Therefore, we report the calculation of a particle propagator and discussion of the corresponding spectral weight in a separate paper.

To outline this paper, the basic properties of a spin Calogero–Sutherland model are shown in section 2. In section 3, the mapping of the field annihilation operator is considered. Using the result of section 3, the hole propagator of a spin Calogero–Sutherland model is derived in section 4. The expression for the hole propagator is rewritten in terms of quasi-hole rapidities in section 5. The properties of the spectral function for $\lambda = 1$ are discussed in section 6.

2. Basic properties

In this section, we review the basic properties of a spin Calogero–Sutherland model.

2.1. Hamiltonian and eigenfunctions

The spin Calogero–Sutherland model is a one-dimensional quantum model which consists of N particles with spin degrees of freedom, moving along the circle of perimeter L. Each pair of particles has an interaction of inverse-square-type potential. Hamiltonian is given by

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{2\pi^2}{L^2} \sum_{i<j} \frac{\lambda(\lambda + P_{ij})}{\sin^2[\pi(x_i - x_j)/L]}, \tag{1}$$

where x_i is the coordinate of ith particle, and P_{ij} is the spin exchange operator for particles i and j. The Hamiltonian (1) has one parameter, an interaction parameter λ, which controls whole physical properties of the system. Physically, $\lambda < 0$ is unrealistic due to the collapse of the particles. In the following sections of this paper, we consider particles with spin 1/2, and λ is restricted to be a non-negative integer for simplicity.

Let the particles be bosons for odd λ and fermions for even λ, following the earlier works on the hole propagator [41, 42], and the boundary condition is chosen to be periodic. In the ground state, $N/2$ is taken to be odd (even) when λ is even (odd) to avoid the degeneracy of the ground state. Introducing new variables $z = \{z_1, \ldots, z_N\}$ with $z_i = \exp[2\pi \mathrm{i} x_i/L]$, the wavefunction $\Psi(\{x_i\}, \{\sigma_i\})$ for the spin 1/2 model is written as the product $\Phi(z, \sigma)\Psi_{0,N}(z)$ of Jastrow type wavefunction

$$\Psi_{0,N}(z) = \prod_i z_i^{-\lambda(N-1)/2} \prod_{i<j} (z_i - z_j)^\lambda \tag{2}$$

and a function $\Phi(z, \sigma)$ of complex spatial coordinates z and spin variables $\sigma = \{\sigma_1, \ldots, \sigma_N\}$. The spin coordinate σ_i takes 1 (2) for spin up (spin down).

When λ is even, particles are fermions and the Jastrow wavefunction $\Psi_{0,N}(z)$ is symmetric with respect to the interchange of z_i and z_j. $\Phi(z, \sigma)$ then obeys the fermionic Fock condition

$$\Phi(\ldots, z_i, \sigma_i, \ldots, z_j, \sigma_j, \ldots) = -\Phi(\ldots, z_j, \sigma_j, \ldots, z_i, \sigma_i, \ldots). \tag{3}$$

When λ is odd, on the other hand, the particles are bosons and the Jastrow wavefunction $\Psi_{0,N}(z)$ is anti-symmetric with respect to the interchange of z_i and z_j. Thus, $\Phi(z, \sigma)$ obeys the fermionic Fock condition (3).

The Jastrow wavefunction $\Psi_{0,N}$ is periodic under the translation $x_i \to x_i + L$ when λ is even or N is odd and anti-periodic otherwise. It thus follows that

$$\Phi(z, \sigma) \text{ is } \begin{cases} \text{periodic} & \text{when } \lambda \text{ is even or } N \text{ is odd} \\ \text{anti-periodic} & \text{otherwise.} \end{cases} \tag{4}$$

A basis for wavefunction satisfying (3) and (4) is given by the Slater determinant of free fermions with spin 1/2:

$$u_{\kappa,\alpha} = \text{Asym}\left[\prod_{i=1}^N z^{\kappa_i} \varphi_{\alpha_i}(\sigma_i)\right] \tag{5}$$

for a set of momenta $\kappa = (\kappa_1, \ldots, \kappa_N)$ and spin configuration $\alpha = (\alpha_1, \ldots, \alpha_N)$. Here, $\alpha_i = 1, 2$ means z component of spin of ith particle being +1/2 and −1/2, respectively. Here, 1-particle spin function $\varphi_{\alpha_i}(\sigma_i)$ is given by $\delta_{3/2-\alpha_i,\sigma_i}$. The symbol Asym[· · ·] means the anti-symmetrization of the function of z and σ:

$$\text{Asym} f(z_1, \sigma_1, \ldots, z_N, \sigma_N) = \sum_{P \in S_N} (-1)^P f(z_{P(1)}, \sigma_{P(1)}, \ldots, z_{P(N)}, \sigma_{P(N)}), \tag{6}$$

where $(-1)^P$ denotes the sign of the permutation P in the symmetric group S_N.

When the basis function (5) obeys the periodic boundary condition, the set of momenta κ belongs to

$$\mathcal{L}_{N,2} = \{\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_N) \in \mathcal{L}_N | \forall\, s \in \mathbf{Z}, \sharp\{\kappa_i \mid \kappa_i = s\} \leqslant 2\}, \tag{7}$$

which is a subset of

$$\mathcal{L}_N = \{\kappa = (\kappa_1, \ldots, \kappa_N) \in \mathbf{Z}^N | \kappa_i \geqslant \kappa_{i+1} \text{ for } i \in [1, N-1]\}. \tag{8}$$

We call the elements of $\mathcal{L}_N$ by shifted partitions. When the basis function (5) obeys the anti-periodic boundary condition, κ belongs to $\mathcal{L}'_{N,2}$, which is defined by

$$\mathcal{L}'_{N,2} = \{\kappa | \kappa + 1/2 \equiv (\kappa_1 + 1/2, \ldots, \kappa_N + 1/2) \in \mathcal{L}_{N,2}\}. \tag{9}$$

For a spin 1/2 system, definition (7) comes from the fact that a single-orbital state can accommodate at most two particles. Furthermore, a pair of particles with the same momentum $\kappa_i = \kappa_{i+1}$ cannot have the same spin state, i.e., $\alpha_i \neq \alpha_{i+1}$ owing to the Pauli exclusion principle. For a given set of momenta $\kappa \in \mathcal{L}_{N,2}$ or $\mathcal{L}'_{N,2}$, therefore, each spin configuration is specified by the element of W_κ defined as

$$W_\kappa = \{\alpha = (\alpha_1, \ldots, \alpha_N) \in [1, 2]^N | \alpha_i < \alpha_{i+1} \text{ if } \kappa_i = \kappa_{i+1}\}. \tag{10}$$

For $N = 2$, W_κ is given by $\{(1,1),(1,2),(2,1),(2,2)\}$ when $\kappa_1 > \kappa_2$, and $\{(1,2)\}$ when $\kappa_1 = \kappa_2$. Thus the basis function $u_{\kappa,\alpha}$ is uniquely specified by $(\kappa,\alpha) \in (\mathcal{L}_{N,2}, W_\kappa)$ under the periodic boundary condition and $(\kappa,\alpha) \in (\mathcal{L}'_{N,2}, W_\kappa)$ under the anti-periodic boundary condition.

Now we define the ordering between the basis functions. First we introduce dominance partial order [14]

$$\nu > \mu \quad \Leftrightarrow \quad |\nu| = |\mu| \qquad \text{and} \qquad \forall\, r > 0, \sum_{i=1}^{r} \nu_i > \sum_{i=1}^{r} \mu_i \tag{11}$$

between $\nu, \mu \in \mathcal{L}_{N,2}$ or $\nu, \mu \in \mathcal{L}'_{N,2}$. Next we define the order for spin configurations as

$$\alpha > \alpha' \quad \Leftrightarrow \quad \sum_{i=1}^{N} \alpha_i = \sum_{i=1}^{N} \alpha'_i, \qquad \text{and nonzero } \alpha'_i - \alpha_i \text{ at the least } i \text{ is positive.} \tag{12}$$

For example, spin configurations with $N = 3$ and $\sum_i^N \alpha_i = 4$ are arranged as

$$(2,1,1) < (1,2,1) < (1,1,2). \tag{13}$$

The order of (κ,α) is then defined by

$$(\kappa,\alpha) > (\kappa',\alpha') \quad \Leftrightarrow \quad \kappa > \kappa', \qquad \text{or} \qquad \kappa = \kappa' \text{ and } \alpha > \alpha'. \tag{14}$$

Uglov showed [43] that the excited parts of the eigenfunctions for the Hamiltonian (1) is characterized by (κ,α) as $\Phi_{\kappa,\alpha}(z,\sigma)$, which can be uniquely defined by the following two conditions:

(i) $\Phi_{\kappa,\alpha}(z,\sigma)$ is expanded by $u_{\kappa',\alpha'}$ satisfying $(\kappa',\alpha') \leqslant (\kappa,\alpha)$:

$$\Phi_{\kappa,\alpha}(z,\sigma) = u_{\kappa,\alpha} + \sum_{(\kappa',\alpha')(<(\kappa,\alpha))} a_{(\kappa',\alpha')(\kappa,\alpha)} u_{\kappa',\alpha'}. \tag{15}$$

(ii) Orthogonal with respect to the norm $\langle\cdots\rangle_{N,\lambda}$:

$$\langle \Phi_{\kappa',\alpha'}, \Phi_{\kappa,\alpha}\rangle_{N,\lambda} = 0 \qquad \text{for} \quad (\kappa',\alpha') \neq (\kappa,\alpha), \tag{16}$$

where the scalar product $\langle\cdots\rangle_{N,\lambda}$ is defined by a weighted integral

$$\langle \Phi', \Phi\rangle_{N,\lambda} = \frac{1}{N!}\left[\prod_{i=1}^{N} \oint \frac{\mathrm{d}z_i}{2\pi \mathrm{i} z_i} \sum_{\sigma_i}\right] \prod_{i\neq j}\left(1 - \frac{z_i}{z_j}\right)^{\lambda} \overline{\Phi'(z,\sigma)}\,\Phi(z,\sigma), \tag{17}$$

where $\overline{\Phi(z,\sigma)}$ means the complex conjugate of $\Phi(z,\sigma)$. The scalar product (17) comes from the usual one:

$$\langle \Psi'|\Psi\rangle = \sum_{\sigma_1=\pm 1/2} \cdots \sum_{\sigma_N=\pm 1/2} \int_0^L \mathrm{d}x_1 \cdots \int_0^L \mathrm{d}x_N\, \overline{\Psi'(\{x_i\},\{\sigma_i\})}\,\Psi(\{x_i\},\{\sigma_i\}). \tag{18}$$

The relation

$$\langle \Psi'|\Psi\rangle = N!L^N \langle \Phi', \Phi\rangle_{N,\lambda} \tag{19}$$

holds when $\Psi = \Phi\Psi_{0,N}$ and $\Psi' = \Phi'\Psi_{0,N}$. The eigenenergy of the eigenfunction $\Phi_{\kappa,\alpha}\Psi_{0,N}$ is given by

$$E_N(\kappa) = (2\pi/L)^2 \sum_i^N (\kappa_i + \lambda(N+1-2i)/2)^2\,. \tag{20}$$

For example, the ground state of an N-particle system is specified by [54] κ^0, α^0 with

$$\kappa^0 = \left(\frac{N}{4} - \frac{1}{2}, \frac{N}{4} - \frac{1}{2}, \frac{N}{4} - \frac{3}{2}, \ldots, -\frac{N}{4} + \frac{1}{2}, -\frac{N}{4} + \frac{1}{2}\right) \tag{21}$$

and

$$\alpha^0 = (1, 2, 1, \ldots, 1, 2).$$

The ground state energy $E_N(\mathrm{g})$ is then given by [31, 33]

$$E_N(\mathrm{g}) = (2\pi/L)^2[(1+2\lambda)^2(M^2-1)M/3 + \lambda^2 M/2] \tag{22}$$

with $M = N/2$.

The above properties (15), (16) and (17) are used for the mapping of the eigenfunctions of the spin 1/2 Calogero–Sutherland model in Uglov's method [43]. The polynomials $\Phi_{\kappa,\alpha}(z,\sigma)$ defined by the above two conditions (15) and (16) are known to coincide with the Yangian Gelfand-Zetlin basis [43, 54], while the Yangian Gelfand-Zetlin basis for a spin 1/2 system is originally defined as the simultaneous eigenfunctions of the quantum determinants of Yangian $Y(\mathfrak{gl}_1)$ and $Y(\mathfrak{gl}_2)$ [43, 53, 54]. In the following, therefore, we refer to $\Phi_{\kappa,\alpha}(z,\sigma)$ as the Yangian Gelfand-Zetlin basis.

2.2. *Macdonald polynomials and gl_2-Jack polynomials*

Macdonald polynomials [14] are symmetric polynomials with two parameters. As a limit of Macdonald polynomials, symmetric Jack polynomials and Schur polynomials can be derived. A lot of mathematical relations of Macdonald polynomial are known and provide important formulae of Jack polynomials, which have been utilized in the calculation of the correlation functions of the scalar Calogero–Sutherland model [22–25, 27].

Macdonald polynomials themselves are the eigenfunctions of the Ruijsenaars–Schneider model [59, 60], which is a relativistic generalization of the scalar Calogero–Sutherland model. The dynamical correlation functions of the Ruijsenaars–Schneider model have been calculated using the properties of Macdonald polynomials [61].

As an index which specifies each symmetric polynomial, we define partitions as the set of non-negative integers arranged in a non-increasing order. We denote the set of partitions with length equal to or shorter than N by

$$\Lambda_N = \{\nu = (\nu_1, \nu_2, \ldots, \nu_N) \in \mathbf{Z}^N \mid \nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_N \geqslant 0\}. \tag{23}$$

The monomial symmetric polynomial m_ν with a partition $\nu \in \Lambda_N$ is defined by the symmetrization of a monomial $z^\nu = z_1^{\nu_1} z_2^{\nu_2} \cdots z_N^{\nu_N}$ as

$$m_\nu = \sum_\sigma z_1^{\nu_{\sigma(1)}} z_2^{\nu_{\sigma(2)}} \cdots z_N^{\nu_{\sigma(N)}}, \tag{24}$$

where the sum is taken for all distinct permutations of the elements of ν.

Macdonald polynomial $P_\nu(z; q, t)$ for $\nu \in \Lambda_N$ is uniquely defined by the following two conditions [14]:

(i) $P_\nu(z; q, t)$ is expanded by m_μ satisfying $\mu \leqslant \nu$:

$$P_\nu(z; q, t) = m_\nu + \sum_{\mu(<\nu)} v_{\nu\mu} m_\mu. \tag{25}$$

(ii) Orthogonal with respect to the norm $\langle\cdots\rangle_{N,q,t}$:

$$\langle P_\mu(z; q, t), P_\nu(z; q, t)\rangle_{N,q,t} = 0 \qquad \text{for} \quad \mu \neq \nu, \tag{26}$$

where the norm of (26) is defined by a weighted integral through the function $(x;q)_\infty = \prod_{r=0}^{\infty}(1-xq^r)$:

$$\langle f,g\rangle_{N,q,t} = \frac{1}{N!}\left[\prod_{i=1}^{N}\oint\frac{dz_i}{2\pi i z_i}\right]\prod_{i\neq j}\frac{(z_i/z_j;q)_\infty}{(tz_i/z_j;q)_\infty}\overline{f(z)}g(z). \tag{27}$$

Since the symmetric Jack polynomial has the limit $t = q^\lambda, q \to 1$ of the Macdonald polynomial, so the above two conditions also define symmetric Jack polynomial uniquely by taking the limit $t = q^\lambda, q \to 1$ of norm (27). Schur symmetric polynomials $s_\nu(z)$ for $\nu \in \Lambda_N$, where

$$s_\nu(z) = \frac{\mathrm{Asym}\left[z_1^{\nu_1+N-1}z_2^{\nu_2+N-2}\cdots z_N^{\nu_N}\right]}{\mathrm{Asym}\left[z_1^{N-1}z_2^{N-2}\cdots z_N^{0}\right]}, \tag{28}$$

can also be obtained as the limit $t = q \to 1$ of $P_\nu(z;q,t)$. Uglov utilized the properties of Macdonald polynomials to calculate the dynamical correlation functions by mapping the Yangian Gelfand-Zetlin basis to symmetric polynomials that are a limit of Macdonald polynomials [43]. These new polynomials are called gl_2-Jack polynomials [43, 44], and defined as

$$P_\nu^{(2\lambda+1)}(z) = \lim_{q=-p,t=-p^{2\lambda+1},p\to 1} P_\nu(z). \tag{29}$$

In the following, we call the limit in (29) the 'Uglov limit'. From (25), (26) and (29), it follows that

(i) $P_\nu^{(2\lambda+1)}(z)$ is expanded by m_μ satisfying $\mu \leqslant \nu$:

$$P_\nu^{(2\lambda+1)}(z) = m_\nu + \sum_{\mu(<\nu)} c_{\nu\mu}m_\mu. \tag{30}$$

(ii) Orthogonal with respect to the scalar product $\{\cdots\}_{N,\lambda}$:

$$\left\{P_\mu^{(2\lambda+1)}, P_\nu^{(2\lambda+1)}\right\}_{N,\lambda} = 0 \qquad \text{for} \quad \mu \neq \nu. \tag{31}$$

The scalar product in (31) is defined as

$$\{f,g\}_{N,\lambda} = \frac{1}{N!}\left[\prod_{i=1}^{N}\oint\frac{dz_i}{2\pi i z_i}\right]\prod_{i\neq j}\left(1-\frac{z_i}{z_j}\right)^{\lambda+1}\left(1+\frac{z_i}{z_j}\right)^{\lambda}\overline{f(z)}g(z), \tag{32}$$

which comes from a limit of (27).

The two properties (30) and (31) can be regarded as the defining properties of gl_2-Jack polynomials. Alternatively, we can define gl_2-Jack polynomials by

$$P_\nu^{(2\lambda+1)}(z) = s_\nu + \sum_{\mu(<\nu)} C_{\nu\mu}s_\mu \tag{33}$$

and (31), because (33) and (30) are equivalent as shown below.

Schur polynomials, which are a limit of Macdonald polynomials, can be expanded by monomial symmetric polynomials, and conversely monomial symmetric polynomials are written in the form of

$$m_\nu(z) = s_\nu + \sum_{\nu'(<\nu)} a_{\nu\nu'}s_{\nu'}, \tag{34}$$

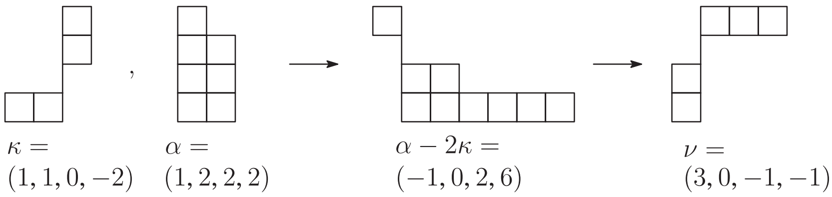

Figure 1. An example of the transformation (37) for $N = 4$. While $(\kappa, \alpha) \in (\mathcal{L}_{N,2}, W_\kappa)$ or $(\mathcal{L}'_{N,2}, W_\kappa)$, $\alpha - 2\kappa$ is a shifted partition of the spinless fermionic system in reverse order, and ν is a shifted partition of the spinless bosonic system.

from which

$$\begin{aligned} P_\nu^{(2\lambda+1)}(z) &= m_\nu + \sum_{\mu(<\nu)} c_{\nu\mu} m_\mu \\ &= s_\nu + \sum_{\nu'(<\nu)} a_{\nu\nu'} s_{\nu'} + \sum_{\mu(<\nu)} c_{\nu\mu}\left(s_\mu + \sum_{\mu'(<\mu)} a_{\mu\mu'} s_{\mu'}\right) \\ &= s_\nu + \sum_{\mu(<\nu)} C_{\nu\mu} s_\mu \end{aligned} \tag{35}$$

follows.

So far Macdonald polynomials P_ν and gl$_2$-Jack polynomials $P_\nu^{(2\lambda+1)}$ have been defined for a partition $\nu \in \Lambda_N$. However, it is convenient to extend the definition of P_ν and $P_\nu^{(2\lambda+1)}$ for ν that belongs to $\mathcal{L}_N$ (or $\mathcal{L}'_N$). When $\nu \in \mathcal{L}_N$ is given by

$$\nu = \mu - J = (\mu_1 - J, \ldots, \mu_N - J)$$

with an integer (half odd integer) J and a partition $\mu \in \Lambda_N$, we define P_ν and $P_\nu^{(2\lambda+1)}$ as

$$P_\nu(z) \equiv (z_1 \cdots z_N)^{-J} P_\mu(z), \qquad P_\nu^{(2\lambda+1)}(z) \equiv (z_1 \cdots z_N)^{-J} P_\mu^{(2\lambda+1)}(z), \tag{36}$$

respectively.

2.3. *Mapping from the Yangian Gelfand-Zetlin basis to gl$_2$-Jack polynomials*

Uglov defined [43] the linear mapping Ω between the set of functions spanned by $u_{\kappa,\alpha}$ with $(\kappa, \alpha) \in (\mathcal{L}_{N,2}, W_\kappa)$ or $(\mathcal{L}'_{N,2}, W_\kappa)$ and the set of symmetric functions as $\Omega(u_{\kappa,\alpha}) = s_\nu$, where the relation between (κ, α) and ν is given by

$$\nu_i = \alpha_{N+1-i} - 2\kappa_{N+1-i} - N + i \tag{37}$$

for the system of N particles with spin 1/2. An example of this transformation is drawn in figure 1. The properties of Ω are listed as follows:

(i) Isometry. The scalar product is preserved under the mapping Ω. For functions $\Phi'(z, \sigma)$ and $\Phi(z, \sigma)$, the relation

$$\langle \Phi', \Phi \rangle_{N,\lambda} = \{\Omega(\Phi'), \Omega(\Phi)\}_{N,\lambda} \tag{38}$$

holds.

(ii) For any symmetric function $f(z)$ [43], the relation

$$\Omega(f(z_1, \ldots, z_N) u_{\kappa,\alpha}(z, \sigma)) = f\left(z_1^{-2}, \ldots, z_N^{-2}\right) \Omega(u_{\kappa,\alpha}(z, \sigma)) \tag{39}$$

holds.

(iii) The correspondence between Yangian Gelfand-Zetlin basis and gl_2-Jack polynomials

$$\Omega(\Phi_{\kappa,\alpha}) = P_\nu^{(2\lambda+1)}. \tag{40}$$

Taking the mapping Ω for both sides of (15) and (16) in the conditions that specify the Yangian Gelfand-Zetlin basis, there appear the defining relations of gl_2-Jack polynomial $P_\nu^{(2\lambda+1)}(z)$ (33) and (31), and consequently, the property (iii) follows. The mapping Ω can be interpreted as a transformation from a multi-component system to a single-component system.

3. Transformation of the field annihilation operator by the mapping

In this section, we consider the mapping of the field annihilation operator. In calculating density the correlation function and the spin correlation function, the spin operator and the density operator are expressed as power sum polynomials, that is, c-numbers [43]. The annihilation and the creation operators in 1-particle Green's function, however, are not the case. We first make sure that the sum of the annihilation operators of the spin ↑ and ↓ on the multi-component model are mapped to the annihilation operator on a single-component model.

3.1. Action of annihilation operator

Generally, the action of the field annihilation operator on a wavefunction is implemented by fixing the coordinate of one of the particles in the wavefunction to that of the annihilation operator as

$$\psi(x)\Psi(x_1,\ldots,x_{N-1},x_N) = \sqrt{N}\zeta^{N-1}\Psi(x_1,\ldots,x_{N-1},x) \tag{41}$$

for the single-component model, where $\zeta = 1$ for boson and $\zeta = -1$ for fermion. Moreover, for a multi-component model, the action of the field annihilation operator with spatial coordinate x and spin coordinate $\sigma\,(=1/2, -1/2)$ is given by

$$\psi_\sigma(x)\Psi(x_1,\sigma_1,\ldots,x_{N-1},\sigma_{N-1},x_N,\sigma_N) = \sqrt{N}\zeta^{N-1}\Psi(x_1,\sigma_1,\ldots,x_{N-1},\sigma_{N-1},x,\sigma). \tag{42}$$

In the following, we write $\psi_\uparrow(x)$ $(\psi_\downarrow(x))$ instead of $\psi_{1/2}(x)$ $(\psi_{-1/2}(x))$ for notational convenience. First, we consider the action of $(\psi_\uparrow(0,0) + \psi_\downarrow(0,0))$ on the wavefunction of a spin Calogero–Sutherland model

$$\Psi(x,\sigma) = \Phi(z,\sigma)\Psi_{0,N}(z), \tag{43}$$

where $\Psi_{0,N}(z)$ is the Jastrow-type wavefunction defined in (2). We can take the action of one of the annihilation operators by restricting the spin configuration of the Yangian Gelfand-Zetlin basis of the both sides of the annihilation operator when calculating the matrix element. The sum of the field operators acts on the wavefunction (43) as

$$\begin{aligned}(\psi_\uparrow(0,0)+\psi_\downarrow(0,0))\Psi &= \zeta^{N-1}\sqrt{N}\sum_{\sigma_N=1,2}\Phi(z_1,\sigma_1,\ldots,z_{N-1},\sigma_{N-1},z_N=1,\sigma_N)\\ &\quad\times\Psi_{0,N}(z_1,\ldots,z_{N-1},z_N=1)\\ &= \zeta^{N-1}\sqrt{N}\sum_{\sigma_N=1,2}\Phi(z_1,\sigma_1,\ldots,z_{N-1},\sigma_{N-1},z_N=1,\sigma_N)\\ &\quad\times\prod_{i=1}^{N-1} z_i^{-\lambda/2}(z_i-1)^\lambda\Psi_{0,N-1}(z_1,\ldots,z_{N-1}).\end{aligned} \tag{44}$$

Next, we consider the action of the field annihilation operator on the wavefunction

$$f(z)\tilde{\Psi}_{0,N}(z), \tag{45}$$

where $f(z)$ is a symmetric function of z, and $\tilde{\Psi}_{0,N}$ is given by

$$\tilde{\Psi}_{0,N}(z) = \prod_{i=1}^{N} z_i^{-(2\lambda+1)(N-1)/2} \prod_{i<j}(z_i - z_j)^{\lambda+1}(z_i + z_j)^{\lambda}. \tag{46}$$

This function is the Uglov limit ($q = -p, t = -p^{2\lambda+1}, p \to 1$) of the ground state wavefunction for the Ruijsenaars–Schneider model [59, 60]. Acting the field annihilation operator on (45), we obtain

$$\begin{aligned}\psi(0,0)f(z)\tilde{\Psi}_{0,N}(z) &= \zeta^{N-1}\sqrt{N} f(z_1,\dots,z_{N-1},z_N=1)\tilde{\Psi}_{0,N}(z_1,\dots,z_{N-1},z_N=1)\\ &= \zeta^{N-1}\sqrt{N} f(z_1,\dots,z_{N-1},z_N=1)\\ &\quad\times\prod_{i=1}^{N-1} z_i^{-\lambda-1/2}(z_i-1)^{\lambda+1}(z_i+1)^{\lambda}\tilde{\Psi}_{0,N-1}(z_1,\dots,z_{N-1}).\end{aligned} \tag{47}$$

We introduce $\tilde{\psi}_s(0,0)$ and $\tilde{\psi}(0,0)$ through the relations

$$\tilde{\psi}_s(0,0)\Phi \equiv (\Psi_{0,N-1})^{-1}\psi_s(0,0)\Phi\Psi_{0,N}, \qquad s = \uparrow \text{ or } \downarrow \tag{48}$$

$$\tilde{\psi}(0,0)f \equiv (\tilde{\Psi}_{0,N-1})^{-1}\psi(0,0)f\tilde{\Psi}_{0,N}. \tag{49}$$

In terms of (48) and (49), (44) and (47) are rewritten as

$$(\tilde{\psi}_\uparrow(0,0)+\tilde{\psi}_\downarrow(0,0))\Phi = \zeta^{N-1}\sqrt{N}\sum_{\sigma_N=1,2}\Phi(z_1,\sigma_1,\cdots,z_{N-1},\sigma_{N-1},z_N=1,\sigma_N)$$

$$\times\prod_{i=1}^{N-1} z_i^{-\lambda/2}(z_i-1)^{\lambda} \tag{50}$$

and

$$\tilde{\psi}(0,0)f(z) = \zeta^{N-1}\sqrt{N} f(z_1,\dots,z_{N-1},z_N=1)\prod_{i=1}^{N-1} z_i^{-\lambda-1/2}(z_i-1)^{\lambda+1}(z_i+1)^{\lambda}, \tag{51}$$

respectively.

3.2. Transformation of the field annihilation operator

In this subsection, we show that

$$\Omega((\tilde{\psi}_\uparrow(0,0)+\tilde{\psi}_\downarrow(0,0))\Phi) = (-1)^{(N-1)\lambda}\left(\prod_{i=1}^{N-1} z_i^{1/2}\right)\tilde{\psi}(0,0)\Omega(\Phi). \tag{52}$$

To derive (52), it suffices to show

$$\Omega((\tilde{\psi}_\uparrow(0,0)+\tilde{\psi}_\downarrow(0,0))u_{\kappa,\alpha}) = (-1)^{(N-1)\lambda}\left(\prod_{i=1}^{N-1} z_i^{1/2}\right)\tilde{\psi}(0,0)s_\nu \tag{53}$$

with (37). This is because (i) Ω is a linear operator, (ii) $u_{\kappa,\alpha}$ is a basis of Φ and (iii) $\Omega(u_{\kappa,\alpha}) = s_\nu$.

Since $u_{\kappa,\alpha}$ is a Slater determinant of N free fermions with spin $1/2$, $u_{\kappa,\alpha}$ with one of the coordinate fixed can be expanded by the Slater determinants of $N-1$ free fermions with spin $1/2$ as

$$\sum_{\sigma=1,2} u_{\kappa,\alpha}(z_1,\sigma_1,\ldots,z_N=1,\sigma) = \sum_{\sigma=1,2}\begin{vmatrix} z_1^{\kappa_1}\varphi_{\alpha_1}(\sigma_1) & \ldots & z_1^{\kappa_N}\varphi_{\alpha_N}(\sigma_1) \\ \vdots & \ddots & \vdots \\ z_{N-1}^{\kappa_1}\varphi_{\alpha_1}(\sigma_{N-1}) & \ldots & z_{N-1}^{\kappa_N}\varphi_{\alpha_N}(\sigma_{N-1}) \\ \delta_{\alpha_1\sigma} & \ldots & \delta_{\alpha_N\sigma}\end{vmatrix}$$
$$= \sum_{i=1}^{N}(-1)^i u_{\ldots,\kappa_{i-1},\kappa_{i+1},\ldots,\cdots,\alpha_{i-1},\alpha_{i+1},\ldots}(z_1,\sigma_1,\ldots,z_{N-1},\sigma_{N-1}). \tag{54}$$

Therefore,

$$\Omega((\tilde{\psi}_\uparrow(0,0)+\tilde{\psi}_\downarrow(0,0))u_{\kappa,\alpha}) = \Omega\left(\zeta^{N-1}\sqrt{N}\sum_{\sigma_N=1,2} u_{\kappa,\alpha}(\{z_i,\sigma_i\})|_{z_N=1}\prod_{i=1}^{N-1} z_i^{-\lambda/2}(z_i-1)^\lambda\right)$$
$$= \zeta^{N-1}\sqrt{N}\prod_{i=1}^{N-1} z_i^\lambda(z_i^{-2}-1)^\lambda\sum_{i=1}^{N}(-1)^i$$
$$\times\,\Omega\left(u_{\ldots,\kappa_{i-1},\kappa_{i+1},\ldots,\ldots,\alpha_{i-1},\alpha_{i+1},\ldots}(z_1,\sigma_1,\ldots,z_{N-1},\sigma_{N-1})\right)$$
$$= \zeta^{N-1}\sqrt{N}\prod_{i=1}^{N-1} z_i^\lambda(z_i^{-2}-1)^\lambda\sum_{i=1}^{N}(-1)^i$$
$$\times\, s_{\nu_1+1,\ldots,\nu_{N-i}+1,\nu_{N-i+2},\ldots,\nu_N}(z_1,\ldots,z_{N-1}). \tag{55}$$

Here, we use property (39) of Ω. We note that though ν is defined by $\nu_i = \alpha_{N+1-i} - 2\kappa_{N+1-i} - N + i$ for an N-particle system, the partition for the Schur polynomial on the right-hand side of (55) has $N-1$ elements.

Since the Schur polynomial is defined by the Slater determinant of spinless fermion, the right-hand side of (55) can be described by a Schur polynomial of N variables with one of the variables fixed as

$$s_\nu(z_1,\ldots,z_{N-1},z_N=1) = \frac{\mathrm{Asym}\left[z_1^{\nu_1+N-1}\cdots z_{N-1}^{\nu_{N-1}+1}1^{\nu_N}\right]}{\mathrm{Asym}\left[z_1^{N-1}\cdots z_{N-1}^{1}1^{0}\right]}$$
$$= \begin{vmatrix} z_1^{\nu_1+N-1} & \ldots & z_1^{\nu_N} \\ \vdots & \ddots & \vdots \\ z_{N-1}^{\nu_1+N-1} & \ldots & z_{N-1}^{\nu_N} \\ 1 & \ldots & 1\end{vmatrix}\cdot\left[\prod_{i=1}^{N-1}(z_i-1)\prod_{1\leqslant i<j\leqslant N-1}(z_i-z_j)\right]^{-1}$$
$$= \frac{\sum_{i=1}^{N}(-1)^i\,\mathrm{Asym}\left[\cdots z_{i-1}^{\nu_{i-1}+N-(i-1)}z_i^{\nu_{i+1}+N-(i+1)}\cdots\right]}{\mathrm{Asym}\left[z_1^{N-2}\cdots z_{N-1}^{0}\right]}\prod_{i=1}^{N-1}(z_i-1)^{-1}$$
$$= \sum_{i=1}^{N}(-1)^i s_{\nu_1+1,\ldots,\nu_{N-i}+1,\nu_{N-i+2},\ldots,\nu_N}(z_1,\ldots,z_{N-1})\prod_{i=1}^{N-1}(z_i-1)^{-1}. \tag{56}$$

Thus, the right-hand side of (55) is rewritten as

$$\text{rhs of (55)} = \zeta^{N-1}(-1)^{(N-1)\lambda}\sqrt{N}\prod_{i=1}^{N-1} z_i^{-\lambda}(z_i-1)^{\lambda+1}(z_i+1)^{\lambda}s_\nu(z_1,\ldots,z_{N-1},z_N=1)$$
$$= (-1)^{(N-1)\lambda}\left(\prod_{i=1}^{N-1} z_i^{1/2}\right)\tilde{\psi}(0,0)s_\nu(z_1,\ldots,z_N). \tag{57}$$

From (55) and (57), relation (53) follows. Using (52), the matrix element of a field annihilation operator is obtained as

$$\langle\Phi_{\kappa',\alpha'},(\tilde{\psi}_\uparrow(0,0)+\tilde{\psi}_\downarrow(0,0))\Phi_{\kappa,\alpha}\rangle_{N-1,\lambda} = \{\Omega(\Phi_{\kappa',\alpha'}),\Omega((\tilde{\psi}_\uparrow(0,0)+\tilde{\psi}_\downarrow(0,0))\Phi_{\kappa,\alpha})\}_{N-1,\lambda}$$
$$= (-1)^{(N-1)\lambda}\left\{P_{\nu'}^{(2\lambda+1)},\left(\prod_{i=1}^{N-1} z_i^{1/2}\right)\tilde{\psi}(0,0)P_\nu^{(2\lambda+1)}\right\}_{N-1,\lambda}$$
$$= (-1)^{(N-1)\lambda}\left\{P_{\nu'-1/2}^{(2\lambda+1)},\tilde{\psi}(0,0)P_\nu^{(2\lambda+1)}\right\}_{N-1,\lambda}. \tag{58}$$

Here, we note that the relation between ν' and (κ',α') is defined by (37) with the number of particles being $N-1$, while the relation between ν and (κ,α) is that with N.

4. Combinatorial description of hole propagator

A hole propagator is one of the 1-particle Green's functions defined as

$$G^-(x,t) = \frac{\langle \mathrm{g},N|\psi_\downarrow^\dagger(x,t)\psi_\downarrow(0,0)|\mathrm{g},N\rangle}{\langle \mathrm{g},N|\mathrm{g},N\rangle}. \tag{59}$$

Here, $|\mathrm{g},N\rangle$ is the state vector of N-particle ground state, whose wavefunction is given by $\Phi_{\kappa^0,\alpha^0}\Psi_{0,N}$. The scalar product in (59) is the conventional one (see (18)).

We rewrite (59) in terms of gl_2-Jack polynomial. First of all, a complete set of the state vectors is inserted between the creation operator and the annihilation operator in the numerator of the right-hand side of (59). Next the complex conjugate is taken so as to alter the creation operator to the annihilation operator. Third, we use the relation

$$\langle(\kappa,\alpha),N-1|\tilde{\psi}_\downarrow(0,0)|\mathrm{g},N\rangle = \langle(\kappa,\alpha),N-1|(\tilde{\psi}_\uparrow(0,0)+\tilde{\psi}_\downarrow(0,0))|\mathrm{g},N\rangle \tag{60}$$

when z component of the total spin S_z of the state (κ,α) is larger than that of the ground state by 1/2. The symbol $|(\kappa,\alpha),N-1\rangle$ denotes the state vector of $(N-1)$-particle state whose wavefunction is $\Phi_{\kappa,\alpha}\Psi_{0,N-1}$. The scalar product between the state vectors is then represented by the scalar product between functions of the Yangian Gelfand-Zetlin basis. Finally, we perform the mapping Ω on the wavefunction and obtain

$$G^-(x,t) = \sum_{\substack{\kappa,\alpha\\ \text{s.t.} S^z_{\text{tot}}=+1/2}} \mathrm{e}^{-\mathrm{i}\omega_\kappa t+\mathrm{i}P_\kappa x}\frac{|\langle(\kappa,\alpha),N-1|(\tilde{\psi}_\downarrow(0,0)+\tilde{\psi}_\uparrow(0,0))|\mathrm{g},N\rangle|^2}{\langle(\kappa,\alpha),N-1|(\kappa,\alpha),N-1\rangle\cdot\langle\mathrm{g},N|\mathrm{g},N\rangle}$$
$$= \frac{1}{LN}\sum_{\substack{\kappa,\alpha\\ \text{s.t.} S^z_{\text{tot}}=+1/2}} \mathrm{e}^{-\mathrm{i}\omega_\kappa t+\mathrm{i}P_\kappa x}\frac{|\langle\Phi_{\kappa,\alpha},(\tilde{\psi}_\downarrow(0,0)+\tilde{\psi}_\uparrow(0,0))\Phi_{\mathrm{g}}\rangle_{N-1,\lambda}|^2}{\langle\Phi_{\kappa,\alpha},\Phi_{\kappa,\alpha}\rangle_{N-1,\lambda}\langle\Phi_{\mathrm{g}},\Phi_{\mathrm{g}}\rangle_{N,\lambda}}$$
$$= \frac{1}{LN}\sum_{\substack{\nu\in\mathcal{L}_{N-1}\\ \text{s.t.} S^z_{\text{tot}}=+1/2}} \mathrm{e}^{-\mathrm{i}\omega_\kappa t+\mathrm{i}P_\kappa x}\frac{\left|\left\{P_{\nu-1/2}^{(2\lambda+1)},\tilde{\psi}(0,0)P_{\mathrm{g}}^{(2\lambda+1)}\right\}_{N-1,\lambda}\right|^2}{\left\{P_\nu^{(2\lambda+1)},P_\nu^{(2\lambda+1)}\right\}_{N-1,\lambda}\left\{P_{\mathrm{g}}^{(2\lambda+1)},P_{\mathrm{g}}^{(2\lambda+1)}\right\}_{N,\lambda}}. \tag{61}$$

The variables of summation ν is related to (κ,α) via

$$\nu_i = \alpha_{N-i} - 2\kappa_{N-i} - N + 1 - i, \qquad i\in[1,N-1]. \tag{62}$$

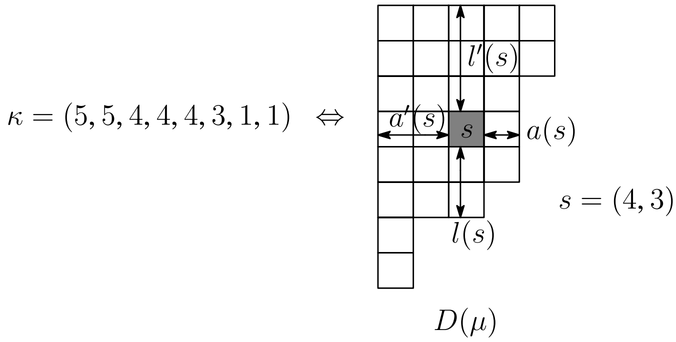

Figure 2. Partition and Young diagram.

The energy ω_κ and momentum P_κ are described in terms of $\kappa \in \mathcal{L}_{N,2}$ or $\mathcal{L}'_{N,2}$ as

$$\begin{cases} \omega_\kappa = E_{N-1}(\kappa) - E_N(\mathrm{g}) \\ P_\kappa = \dfrac{2\pi}{L} \displaystyle\sum_i \kappa_i \end{cases} \tag{63}$$

with

$$E_{N-1}(\kappa) = \left(\frac{2\pi}{L}\right)^2 \sum_i^{N-1} \left(\kappa_i + \frac{\lambda(N-2i)}{2}\right)^2 \tag{64}$$

and (22). Ω maps Φ_{κ^0,α^0} to $P^{(2\lambda+1)}_{\nu^0} \equiv P^{(2\lambda+1)}_{\mathrm{g}}$ with

$$\nu^0 = \left(-\frac{N}{2}+2, -\frac{N}{2}+2, \ldots, -\frac{N}{2}+2\right), \tag{65}$$

and we obtain the gl_2-Jack polynomial of the ground state as $P^{(2\lambda+1)}_{\mathrm{g}}(z) = \prod_i z_i^{-N/2+2}$. The restriction on the sum is considered later.

4.1. Expansion by gl_2-Jack polynomials

The matrix elements of correlation functions in the Sutherland model are expressed in terms of partitions. Partitions can be expressed graphically by Young diagrams, whose correspondence with partition is drawn in figure 2. From the top, κ_1 squares are placed in the first row, and κ_2 squares in the second row, and so on. Each square is specified by two-dimensional coordinates, labeling the square at the upper left by $s = (1, 1)$. The first coordinate indicates the vertical position and the second the horizontal position. The length of a partition is defined by the number of nonzero elements in the partition, and equal to the length of the first column of the Young diagram.

Some relations of symmetric polynomials used in calculating correlation functions are described in terms of the variables defined by the Young diagram, $a(s), a'(s), l(s), l'(s)$ for $s = (i, j)$, as

$$a(s) = \kappa_i - j, \qquad l(s) = \kappa'_j - i,$$
$$a'(s) = j - 1, \qquad l'(s) = i - 1,$$

where κ_i is the ith element of the partition κ, and κ'_j is the length of the j th column in the Young diagram (figure 2).

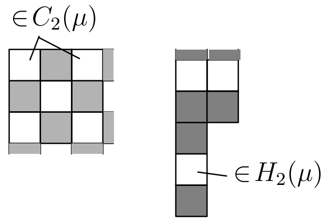

Figure 3. $C_2(\mu)$ and $H_2(\mu)$. The white squares belong to $C_2(\mu)$ and $H_2(\mu)$ respectively.

In calculating the numerator on the right-hand side of (61), the term $\prod_{i=1}^{N-1}(z_i-1)^{\lambda+1}(z_i+1)^{\lambda}$ appears in $\tilde{\psi}(0,0)P_{\mathrm{g}}^{(2\lambda+1)}(z)$ and it can be expanded by gl$_2$-Jack polynomials. The expansion formula is obtained from the corresponding relation of Macdonald polynomials [14]

$$\prod_{i=1}^{N}\frac{(z_i;q)}{(tz_i;q)}=\sum_{\mu\in\Lambda_N}\frac{t^{|\mu|}(t^{-1})_{\mu}^{(q,t)}}{h_{\mu'}(t,q)}P_{\mu}(z:q,t), \tag{66}$$

where

$$\begin{cases} (r)_{\mu}^{(q,t)}=\displaystyle\prod_{s\in D(\mu)}(t^{l'(s)}-q^{a'(s)}r) \\ h_{\mu'}(t,q)=\displaystyle\prod_{s\in D(\mu)}(1-q^{a(s)+1}t^{l(s)}). \end{cases} \tag{67}$$

Taking the limit $q=-p,t=-p^{2\lambda+1},p\to 1$ for (66), we obtain

$$\prod_{i=1}^{N-1}(1-z_i)^{\lambda+1}(1+z_i)^{\lambda}=\sum_{\substack{\mu\in\Lambda_{N-1}\\ \text{s.t.}|C_2(\mu)|+|H_2(\mu)|=|\mu|}}P_{\mu}^{(2\lambda+1)}(z)\cdot(-1)^{|\mu|+\sum l'(s)}$$
$$\times\frac{\prod_{s\in D(\mu)\backslash C_2(\mu)}(a'(s)-(2\lambda+1)(l'(s)+1))}{\prod_{s\in H_2(\mu)}(a(s)+1+(2\lambda+1)l(s))}, \tag{68}$$

where $C_2(\mu)$, $H_2(\mu)$ are the subsets of the squares in the Young diagram $D(\mu)$ defined, respectively, as (figure 3)

$$C_2(\mu)=\{s\in D(\mu)|a'(s)+l'(s)\equiv 0 \bmod 2\} \tag{69}$$

$$H_2(\mu)=\{s\in D(\mu)|a(s)+l(s)+1\equiv 0 \bmod 2\}. \tag{70}$$

The set $A\backslash B$ means the complementary set of B in A. On the right-hand side of (68), the sum of μ is restricted to the partition satisfying $|C_2(\mu)|+|H_2(\mu)|=|\mu|$. The numerator of (61) is written in terms of the variables of the Young diagram and the norm of intermediate states as

$$N\sum_{\substack{\mu\in\Lambda_{N-1}\\ \text{s.t.}|C_2(\mu)|+|H_2(\mu)|=|\mu|}}\delta_{\nu-1/2,\mu-(N/2+\lambda-2)-1/2} \tag{71}$$
$$\times\left(\frac{\prod_{s\in D(\mu)/C_2(\mu)}(a'(s)-(2\lambda+1)(l'(s)+1))}{\prod_{s\in H_2(\mu)}(a(s)+1+(2\lambda+1)l(s))}\right)^2$$
$$\times\left|\left\{P_{\mu}^{(2\lambda+1)},P_{\mu}^{(2\lambda+1)}\right\}_{N-1,\lambda}\right|^2. \tag{72}$$

The norm of gl_2-Jack polynomial is also obtained by that of Macdonald polynomial [14]

$$\left\{P_\mu^{(2\lambda+1)}, P_\mu^{(2\lambda+1)}\right\}_{N,\lambda} = c_N^{(2\lambda+1,2)} \prod_{s\in C_2(\mu)} \frac{a'(s) + (2\lambda+1)(N - l'(s))}{a'(s) + 1 + (2\lambda+1)(N - l'(s) - 1)} \times \prod_{s\in H_2(\mu)} \frac{a(s) + 1 + (2\lambda+1)l(s)}{a(s) + (2\lambda+1)(l(s)+1)}, \tag{73}$$

where $c_N^{(2\lambda+1,2)}$ is the norm of the ground state of the system of N particles with spin degrees of freedom 2:

$$c_N^{(2\lambda+1,2)} = \prod_{1\leqslant i<j\leqslant N} C^{(2\lambda+1)}(j-i), \tag{74}$$

$$C^{(2\lambda+1)}(k) = \begin{cases} \dfrac{\Gamma\left(\frac{(2\lambda+1)(k+1)}{2}\right)\Gamma\left(\frac{(2\lambda+1)(k-1)}{2}+1\right)}{\Gamma\left(\frac{(2\lambda+1)k}{2}+\frac{1}{2}\right)^2} & k = 1 \bmod 2 \\ \dfrac{\Gamma\left(\frac{(2\lambda+1)(k+1)}{2}+\frac{1}{2}\right)\Gamma\left(\frac{(2\lambda+1)(k-1)}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{(2\lambda+1)k}{2}\right)\Gamma\left(\frac{(2\lambda+1)k}{2}+1\right)} & k = 0 \bmod 2. \end{cases} \tag{75}$$

To summarize this section, the hole propagator is written in terms of the variables defined by the Young diagram as

$$G^-(x,t) = \frac{\Gamma((\lambda+1/2)N-\lambda)\Gamma(\lambda+1)}{\Gamma((\lambda+1/2)N)L} \times \sum_{\substack{\mu\in\Lambda_{N-1},\text{s.t.}S^z_{\text{tot}}=+1/2 \\ |C_2(\mu)|+|H_2(\mu)|=|\mu|}} \mathrm{e}^{-\mathrm{i}\tilde{\omega}_\mu t+\mathrm{i}\tilde{P}_\mu x} \frac{X_\mu^2 Y_\mu(0)}{Y_\mu(\alpha-1/2)Z_\mu(\alpha)Z_\mu(1/2)} \tag{76}$$

with $\alpha = 1/(2(2\lambda+1))$. $\tilde{\omega}_\mu$ and $\tilde{P}_\mu$ are, respectively, given by $\tilde{\omega}_\mu = \omega_\kappa$ and $\tilde{P}_\mu = P_\kappa$ in (63) through the relation $\mu_i = \alpha_{N-i} - 2\kappa_{N-i} - N/2 + \lambda - 1 - i$. The prefactor in front of the summation comes from $c_{N-1}^{(2\lambda+1,2)}\big/(c_N^{(2\lambda+1,2)}L)$. We have introduced the following notations:

$$X_\mu \equiv \prod_{s\in D(\mu)\backslash C_2(\mu)} (-\alpha a'(s) + (l'(s)+1)/2), \tag{77}$$

$$Y_\mu(r) \equiv \prod_{s\in C_2(\mu)} (\alpha a'(s) + r + (N-1-l'(s))/2), \tag{78}$$

$$Z_\mu(r) \equiv \prod_{s\in H_2(\mu)} (\alpha a(s) + r + l(s)/2). \tag{79}$$

4.2. Restrictions on μ

Combining the condition on ν with that on μ, that is,

(i) z component of the total spin of the intermediate state is larger than that of the ground state added by 1/2,

(ii) $|C_2(\mu)| + |H_2(\mu)| = |\mu|$,

there are two conditions for μ to satisfy in the sum. On the other hand, the factor $a'(s) - (2\lambda+1)(l'(s)+1)$ in the product of (76) implies that the partitions including the square at $(1, 2\lambda+2)$ have no contribution in the sum of (76). Hence, in the following, we

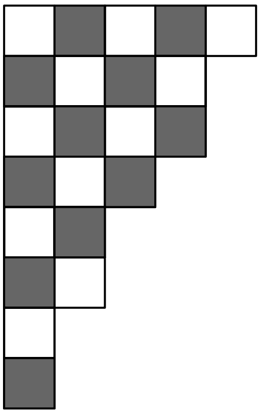

Figure 4. Young diagram of a quasi-hole state for $\lambda = 2$. In this diagram, $\{\mu'_1, \ldots, \mu'_5\} = \{8, 6, 4, 3, 1\}$, $P = \{1, 3, 4\}$ and $Q = \{2, 5\}$. White squares belong to $C_2(\mu)$ and shaded one belong to $D(\mu)\backslash C_2(\mu)$.

consider only the partition with $\mu_1 \leqslant 2\lambda + 1$ and length equal to or shorter than $N - 1$. An example is shown for $\lambda = 2$ in figure 4. We define the subsets of $2\lambda + 1$ columns of the Young diagram by P and Q as follows. The jth column belongs to P when the difference between j and the length of the jth column μ'_j is odd, and Q is the complementary set of P

$$P = \{j \in [1, 2\lambda + 1] | \mu'_j - j : \text{odd}\} \tag{80}$$

$$Q = \{j \in [1, 2\lambda + 1] | \mu'_j - j : \text{even}\}. \tag{81}$$

In figure 4, $\{\mu'_1, \ldots, \mu'_5\} = \{8, 6, 4, 3, 1\}$ and hence $P = \{1, 3, 4\}$ and $Q = \{2, 5\}$.

We define n as the number of the columns in the set Q. Then $|C_2(\mu)|$, $|H_2(\mu)|$ are written in terms of n as

$$|C_2(\mu)| = \frac{|\mu| - \lambda + n}{2} \tag{82}$$

$$|H_2(\mu)| = \frac{|\mu| - n - \lambda(2\lambda + 1) + 2(2\lambda \cdot n - n(n-1))}{2}. \tag{83}$$

These relations are derived in appendix A. From (82) and (83), the condition $|C_2(\mu)| + |H_2(\mu)| = |\mu|$ can be described as the condition on n by

$$|C_2(\mu)| + |H_2(\mu)| = |\mu| \quad \Leftrightarrow \quad n = \lambda, \lambda + 1. \tag{84}$$

Next, z component of the total spin for a partition μ is written as [43]

$$\begin{aligned} S_\mu &= -|C_2(\mu)| + |D_2(\mu)\backslash C_2(\mu)| + \tfrac{1}{2} \\ &= \lambda - n + \tfrac{1}{2}. \end{aligned} \tag{85}$$

Therefore, the condition for total spin is also written in terms of n by

$$S_\mu = 1/2 \quad \Leftrightarrow \quad n = \lambda. \tag{86}$$

Relation (85) gives the meaning of relation (84) that intermediate states μ that arise by acting $(\psi_\downarrow + \psi_\uparrow)$ on the wavefunction of the ground state contain only the states with $S_\mu = +1/2$ or

$-1/2$. Then relation (86) imposes the restriction to take only one of the annihilation operators. Putting together all conditions, the sum of μ is taken over the partitions satisfying

$$\mu_1 \leqslant 2\lambda + 1 \text{ and length equal to or shorter than } N - 1 \tag{87}$$

$$n = \lambda. \tag{88}$$

5. Quasi-hole description of hole propagator

In this section, we rewrite the expression for the hole propagator in terms of rapidities and spins of quasi-holes.

5.1. Rapidities and spins of quasi-holes

From (87), $(N-1)$-particle states relevant to the hole propagator (76) are parameterized by the set of length of each column $\{\mu'_1, \ldots, \mu'_{2\lambda+1}\}$ and the set of 'spin variables'

$$\{\sigma_1, \ldots, \sigma_{2\lambda+1}\},$$

the entry of which is defined by

$$\sigma_j = \begin{cases} 1/2, & j \in P \\ -1/2, & j \in Q \end{cases} \tag{89}$$

for $i \in [1, 2\lambda+1]$. For later convenience, we introduce auxiliary notations $\mu'_0 = N-1$, $\mu'_{2\lambda+2} = 0$ and

$$\sigma_0 = 1/2, \qquad \sigma_{2\lambda+2} = -1/2. \tag{90}$$

Regarding the length μ'_0 of '0th' column as $N - 1$, which is odd and that of $2\lambda + 2$th as 0, definition (90) is a natural extension of (89).

Furthermore, we introduce the renormalized momentum

$$\tilde{\mu}'_j = \mu'_j - \frac{N-1}{2} + \frac{\lambda + 1 - j}{2\lambda + 1} \tag{91}$$

for $j \in [0, 2\lambda+2]$. In terms of (89) and (91) for $j \in [1, 2\lambda+1]$, excitation energy of eigenstates relevant to the hole propagator is written in the form of free particles. The matrix element appearing in the hole propagator is written in terms of $\{\tilde{\mu}'_j, \sigma_j\}$ for $j \in [0, 2\lambda + 2]$. We will see in the following that $\tilde{\mu}'_j + \sigma_j$ and σ_j can be interpreted as the rapidity and spin of jth quasi-hole, respectively.

5.2. Hole propagator in finite-sized systems

The excitation energy $\tilde{\omega}_\mu \equiv \omega_\kappa$ and the momentum $\tilde{P}_\mu \equiv P_\kappa$ are described by

$$\tilde{\omega}_\mu = -(2\lambda + 1) \sum_{j=1}^{2\lambda+1} \left(\frac{\pi(\tilde{\mu}'_j + \sigma_j)}{L} \right)^2 + \frac{4\pi^2 \lambda(\lambda + 1)}{3L^2} \tag{92}$$

$$\tilde{P}_\mu = (2\pi/L) \sum_{i=1}^{N-1} \kappa_i = -\sum_{j=1}^{2\lambda+1} (\pi(\tilde{\mu}'_j + \sigma_j)/L). \tag{93}$$

The derivation of (92) and (93) is given in appendix B. The matrix element in the hole propagator (76) can be described in terms of the renormalized momenta and spin variables as the energy spectrum.

X_μ, $Y_\mu(r)$ and $Z_\mu(r)$ defined in (77), (78) and (79), respectively, are described as

$$X_\mu = \prod_{j=1}^{2\lambda+1} \frac{\Gamma[(\tilde{\mu}'_j - \tilde{\mu}'_{2\lambda+2} + 1 - \delta_{\sigma_j\sigma_{2\lambda+2}})/2]}{\Gamma[j/(2\lambda+1)]}, \tag{94}$$

$$Y_\mu(r) = \prod_{j=1}^{2\lambda+1} \frac{\Gamma[\alpha + r + N/2 + (j-1)/(2\lambda+1)]}{\Gamma[(\tilde{\mu}'_0 - \tilde{\mu}'_j + 1 + \delta_{\sigma_0\sigma_j})/2 - \alpha + r]}, \tag{95}$$

$$Z_\mu(r) = (\Gamma[r+1/2])^{-(2\lambda+1)} \prod_{j=1}^{2\lambda+1} \Gamma[(\tilde{\mu}'_j - \tilde{\mu}'_{2\lambda+2} - \delta_{\sigma_j\sigma_{2\lambda+2}})/2 + r - \alpha + 1/2]$$
$$\times \prod_{1\leqslant j<k\leqslant 2\lambda+1} \frac{\Gamma[(\tilde{\mu}'_j - \tilde{\mu}'_k - \delta_{\sigma_j\sigma_k})/2 + r - \alpha + 1/2]}{\Gamma[(\tilde{\mu}'_j - \tilde{\mu}'_k + \delta_{\sigma_j\sigma_k})/2 + r]}, \tag{96}$$

as shown in appendix B. Using (94), (95) and (96), expression (76) is rewritten as

$$G^-(x,t) = K_\lambda(N) d \sum_{0\leqslant\mu'_{2\lambda+1}\leqslant\cdots\leqslant\mu'_1\leqslant N-1} \sum_{\{\sigma_j\}} \delta_{(\sum_j \sigma_j),1/2} \exp[-i\tilde{\omega}_\mu t + i\tilde{P}_\mu x] F(\{\tilde{\mu}'_j, \sigma_j\}), \tag{97}$$

with the particle density $d = N/L$ and the constant $K_\lambda(N)$:

$$K_\lambda(N) = \frac{\Gamma((\lambda+1/2)N - \lambda)\Gamma(\lambda+1)}{\Gamma((\lambda+1/2)N - \lambda)} \frac{(\Gamma[((\lambda+1)/(2\lambda+1))])^{2\lambda+1}}{\prod_{j=1}^{2\lambda+1}(\Gamma[j/(2\lambda+1)])^2}$$
$$\times \prod_{j=1}^{2\lambda+1} \frac{\Gamma[j/(2\lambda+1) + N/2 - \alpha]}{\Gamma[j/(2\lambda+1) + N/2 - 1/2]} \tag{98}$$

and the form factor

$$F(\{\tilde{\mu}'_j, \sigma_j\}) = \prod_{j=1}^{2\lambda+1} \frac{\Gamma[(\tilde{\mu}'_0 - \tilde{\mu}'_j + \delta_{\sigma_j\sigma_0})/2]}{\Gamma[(\tilde{\mu}'_0 - \tilde{\mu}'_j + 1 + \delta_{\sigma_j\sigma_0})/2 - \alpha]}$$
$$\times \prod_{j=1}^{2\lambda+1} \frac{\Gamma[(\tilde{\mu}'_j - \tilde{\mu}'_{2\lambda+2} + 1 - \delta_{\sigma_j\sigma_{2\lambda+2}})/2]}{\Gamma[(\tilde{\mu}'_j - \tilde{\mu}'_{2\lambda+2} + 1 - \delta_{(\sigma_j\sigma_{2\lambda+2})})/2 + 1/2 - \alpha]}$$
$$\times \prod_{1\leqslant j<k\leqslant 2\lambda+1} \frac{\Gamma[(\tilde{\mu}'_j - \tilde{\mu}'_k + \delta_{\sigma_j\sigma_k})/2 + \alpha]\Gamma[(\tilde{\mu}'_j - \tilde{\mu}'_k + \delta_{\sigma_j\sigma_k})/2 + 1/2]}{\Gamma[(\tilde{\mu}'_j - \tilde{\mu}'_k - \delta_{\sigma_j,\sigma_k})/2 + 1/2]\Gamma[(\tilde{\mu}'_j - \tilde{\mu}'_k - \delta_{\sigma_j\sigma_k})/2 + 1 - \alpha]}. \tag{99}$$

5.3. Thermodynamic limit

Changing the variables to new ones by

$$\frac{\mu'_j}{N} = \frac{1-u_j}{2}, \tag{100}$$

which are finite in the thermodynamic limit. Using the relation $\Gamma(N+a)/\Gamma(N+b) \to N^{a-b} (N \to \infty)$, (76) converges to an expression with finite value. The result in the thermodynamic limit $N \to \infty (N/L = \text{const.})$ is written as

$$G^-(x,t) = K'_\lambda d \left(\prod_{i=1}^{2\lambda+1} \sum_{\sigma_i=\pm 1/2} \int_{-1}^{1} \mathrm{d}u_i \right) \delta_{(\sum_{i=1}^{2\lambda+1} \sigma_i),1/2}$$
$$\times \prod_{j=1}^{2\lambda+1} \left(1-u_j^2\right)^{-\lambda/(2\lambda+1)} \prod_{j<k}^{2\lambda+1} |u_k - u_j|^{2\delta_{\sigma_j,\sigma_k} - 2\lambda/(2\lambda+1)}$$
$$\times \exp\left[\mathrm{i} \left((2\lambda+1) \left(\frac{\pi d}{2}\right)^2 \sum_{j=1}^{2\lambda+1} u_j^2 \right) t + \mathrm{i} \left(\frac{\pi d}{2} \sum_{j=1}^{2\lambda+1} u_j \right) x \right], \tag{101}$$

with the overall constant

$$K'_\lambda = \frac{\Gamma(\lambda+1)}{4(2\lambda+1)^\lambda \Gamma(2\lambda+2)} \prod_{j=1}^{2\lambda+1} \frac{\Gamma\left((\lambda+1)/(2\lambda+1)\right)}{\Gamma\left(j/(2\lambda+1)\right)^2}. \tag{102}$$

Result (101) for the interaction parameter λ is similar to the hole propagator of a scalar Calogero–Sutherland model for the interaction parameter $2\lambda + 1$ with respect to the number of quasi-hole, spectrum, form of the matrix element. The exponents of an interparticle part of the matrix element in (101), however, differs from that of the scalar model, and it depends on whether spins of a pair of particles are parallel or anti-parallel, which is specific to a spin-generalized model. The result coincides with the earlier result [41, 42].

6. Spectral function for $\lambda = 1$

The spectral function for a hole propagator is defined as a Fourier transformation of $G^-(x,t)$:

$$A^-(p,\epsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}(\epsilon-\mu)t - \mathrm{i}px} G^-(x,t). \tag{103}$$

We calculate the spectral function at $\lambda = 1$, which is the easiest nontrivial case in this model. The spectral function has non-zero value at a finite area in an energy–momentum plane (figure 5). The area is enclosed by four parabolic lines, and each of them can be interpreted in terms of a quasi-hole picture. The upper edge $\epsilon = -p^2 + 9(\pi d)^2/4$, with $|p| < 3\pi d/2$ corresponds to the excited state with three quasi-holes having the same momentum. The three lower edges

$$\begin{aligned} &\epsilon = -3p^2 + 3(\pi d)^2/4, && |p| < \pi d/2, \\ &\epsilon = -3(p+\pi d)^2 + 3(\pi d)^2/4, && -3\pi d/2 < p < -\pi d/2 \\ &\epsilon = -3(p-\pi d)^2 + 3(\pi d)^2/4, && \pi d/2 < p < 3\pi d/2 \end{aligned}$$

correspond to the states that one quasi-hole is excited, while the other two quasi-holes reside at the Fermi points.

The intensity of the spectral function is also drawn in figure 5. The spectral function diverges at two lower edges $\epsilon = -3(p \pm \pi d)^2 + 3(\pi d)^2/4$ for $\pi d/2 < |p| < 3\pi d/2$ and two other parabolic lines $\epsilon = -3(p \mp \pi d/2)^2/2 + 3(\pi d)^2/2$, with $-\pi d/2 \leqslant p \leqslant 3\pi d/2$ for the upper sign and $-3\pi d/2 \leqslant p \leqslant -\pi d/2$ for the lower sign. At the lower edge $\epsilon = -3p^2 + 3(\pi d)^2/4$ for $-\pi d/2 \leqslant p \leqslant \pi d/2$, the spectral function becomes zero and arises as $(\epsilon + 3p^2 - 3(\pi d)^2/4)^{1/3}$, and at $\epsilon = -3(p \pm \pi d)^2 + 3(\pi d)^2/4$ it diverges as $(\epsilon + 3(p \pm \pi d)^2 - 3(\pi d)^2/4)^{-1/3}$. At the middle line, the spectral function diverges as $|\epsilon + 3(p \mp \pi d/2)^2/2 - 3(\pi d)^2/2|^{-1/6}$, which will be derived in appendix C. At the upper

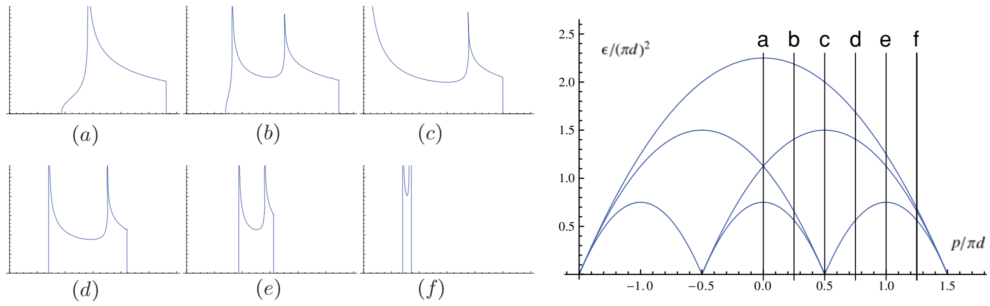

Figure 5. Spectral function of spin 1/2 Calogero–Sutherland model for $\lambda = 1$.

edge, the spectral function takes finite value proportional to $(1 - (2p/3\pi d)^2)^{-1}$. Arikawa pointed out [58] that the hole propagator of $SU(2)$ Calogero–Sutherland model for $\lambda = 1$ is equivalent to the dynamical colour correlation function of the $SU(3)$ Haldane–Shastry model [45, 46, 58]. Although the singularities of the upper edge and lower edges have been obtained in earlier works [46, 58], our calculation shows that the spectral function at the middle line diverges with exponent 1/6 contrary to [46]. We confirm this exponent also by numerical calculation.

7. Conclusion

To calculate the hole propagator of a spin Calogero–Sutherland model by Uglov's method, we transformed the field annihilation operator on the Yangian Gelfand-Zetlin basis by the mapping Ω and proved that it also becomes the field annihilation operator on gl_2-Jack polynomials. This ensures the possibility of calculating the 1-particle Green's function using Uglov's method.

Next, by using this method, we calculated the hole propagator for a non-negative integer interaction parameter by taking the restricted product over the Young diagrams of intermediate states. The thermodynamic limit of the hole propagator was also taken, and we confirmed that the result obtained here coincides with that of the former results with a Jack polynomial with prescribed symmetry. Spectral function for $\lambda = 1$ was calculated, and drawn in the energy–momentum plane. There appear the divergences of intensity on one and two quasi-hole excitation lines.

The calculation of the particle propagator will be published in a separate paper.

Acknowledgments

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Appendix A. Restriction on the product

The number of the squares of the subset of the Young diagram, $|C_2(\mu)|$ and $|H_2(\mu)|$, can be uniquely described in terms of n, the number of the columns which belong to Q. This can be verified by describing $|C_2(\mu)| - |D(\mu)\backslash C_2(\mu)|$ and $|H_2(\mu)| - |D(\mu)\backslash H_2(\mu)|$ in terms of n, where $A\backslash B$ means the complement of B in A.

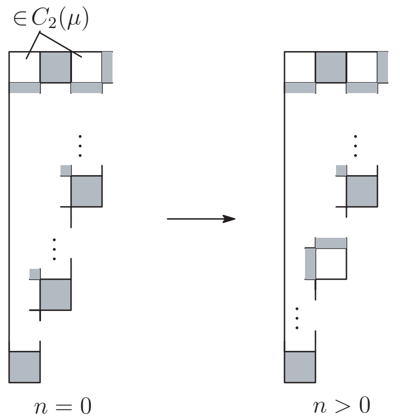

Figure A1. The relation between the number of the squares of $C_2(\mu)$ and $D(\mu)\backslash C_2(\mu)$ and n. White squares are the elements of $C_2(\mu)$, and shaded squares are that of $D(\mu)\backslash C_2(\mu)$. For $n = 0$, the bottom of each column are shaded squares (the elements of the $D(\mu)\backslash C_2(\mu)$), and for $n > 0$, there appears the white square (the elements of the $C_2(\mu)$) in n columns.

In each column and each row of the Young diagram, the element of $C_2(\mu)$ and that of $D(\mu)\backslash C_2(\mu)$ are aligned alternately, and at the upper left of the diagram $s = (1, 1) \in C_2(\mu)$. From the definition of the set P and Q ((80) and (81)), the jth column with $j \in P$ has an even number of squares for odd j and an odd number for even j. Thus the square at the bottom of the jth column in P is an element of $D(\mu)\backslash C_2(\mu)$. For $n = 0$, all the columns belong to the set P, and therefore $|C_2(\mu)| - |D(\mu)\backslash C_2(\mu)| = -\lambda$. The change of a column in P to Q increases $|C_2(\mu) - D(\mu)/C_2(\mu)|$ by 1 (figure A1). We thus obtain

$$|C_2(\mu)| - |D(\mu)\backslash C_2(\mu)| = -\lambda + n. \tag{A.1}$$

Next we describe $|H_2(\mu)| - |D(\mu)\backslash H_2(\mu)|$ in terms of n. First we consider a partition having no adjacent columns with the same length, and then we consider the general case. In each column, the square at the bottom $s = (\mu'_j, j)$ is an element of $D(\mu)\backslash H_2(\mu)$. The element of $H_2(\mu)$ and that of $D(\mu)\backslash H_2(\mu)$ are aligned alternately except for several rows. The exceptions occur at the μ'_kth row and $\mu'_k + 1$th row in the jth column with $k = [j+1, j+2, \ldots, 2\lambda+1]$, where two elements of the same subset $H_2(\mu)$ or $D(\mu)\backslash H_2(\mu)$ are aligned vertically (figure A2). Thus, we remove the μ'_j+1th row with $j = [2, 3, \ldots, 2\lambda+1]$ from the original Young diagram, and make the diagrams $D(\tilde{\mu})$ in which the element of $H_2(\mu)$ and that of $D(\mu)\backslash H_2(\mu)$ are aligned alternately without any exceptions in each column. The removed 2λ rows make a new diagram $D(\Delta\mu)$ with length 2λ and $(\Delta\mu)_1 = 2\lambda$ (figure A2).

For $n = 0$, each column of $D(\tilde{\mu})$ has even number of squares. Therefore, the contribution to $|H_2(\mu)| - |D(\mu)\backslash H_2(\mu)|$ from $\tilde{\mu}$ is given by

$$|H_2(\mu)|_{\tilde{\mu}} - |D(\mu)\backslash H_2(\mu)|_{\tilde{\mu}} = 0. \tag{A.2}$$

When a column in $D(\mu)$ changes from P to Q, the number of columns in $D(\tilde{\mu})$ with odd length increases by one. We thus obtain

$$|H_2(\mu)|_{\tilde{\mu}} - |D(\mu)\backslash H_2(\mu)|_{\tilde{\mu}} = -n \tag{A.3}$$

for $n \geqslant 0$.

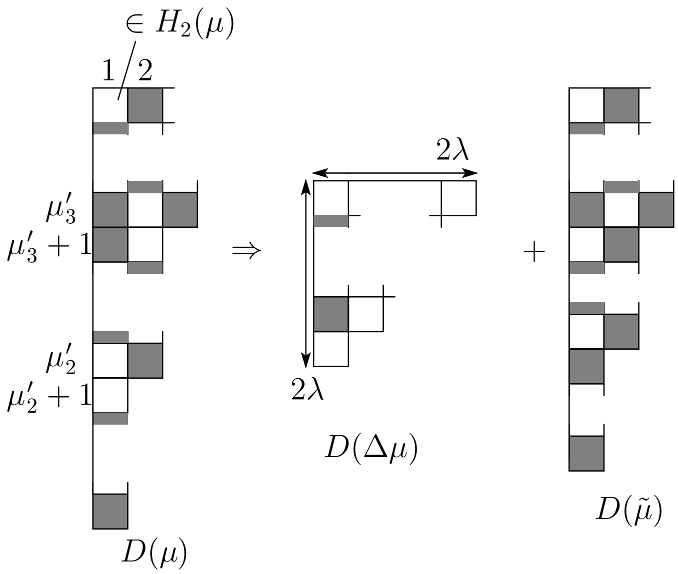

Figure A2. Young diagram $D(\mu)$ is decomposed to two diagrams $D(\Delta\mu)$ and $D(\tilde{\mu})$. White squares are the elements of $H_2(\mu)$, and shaded squares are that of $D(\mu)\backslash H_2(\mu)$. In each column of $D(\tilde{\mu})$, the element of $H_2(\mu)$ and that of $D(\mu)\backslash H_2(\mu)$ are aligned alternately.

Now we consider the contribution of $D(\Delta\mu)$ to $|H_2(\mu)| - |D(\mu)\backslash H_2(\mu)|$. The square $s = (j, k)$ with $1 \leqslant j \leqslant 2\lambda$, $1 \leqslant k \leqslant 2\lambda + 1 - j$ in $D(\Delta\mu)$ comes from $s' = (\mu'_{2\lambda+2-j} + 1, k)$ in $D(\mu)$. Using $a(s') = 2\lambda + 2 - j - k - 1$ and $l(s') = \mu'_k - \mu'_{2\lambda+2-j} - 1$, it follows that

$$a(s') + l(s') + 1 \equiv \begin{cases} 1 & (k, 2\lambda + 2 - j) \in (P, P) \text{ or } (Q, Q) \\ 0 & (k, 2\lambda + 2 - j) \in (P, Q) \text{ or } (Q, P), \end{cases} \tag{A.4}$$

and hence $s = (j, k) \in H_2(\mu)$ in $D(\Delta\mu) \Leftrightarrow (k, 2\lambda + 2 - j) \in (P, Q)$ or (Q, P). Since two coordinates k and $2\lambda + 2 - j$ vary with $1 \leqslant k < 2\lambda + 2 - j \leqslant 2\lambda + 1$,

$$|H_2(\mu)|_{\Delta\mu} = n(2\lambda + 1 - n) \tag{A.5}$$

$$|D(\mu)\backslash H_2(\mu)|_{\Delta\mu} = n(n-1)/2 + (2\lambda + 1 - n)(2\lambda - n)/2 \tag{A.6}$$

and

$$|H_2(\mu)|_{\Delta\mu} - |D(\mu)\backslash H_2(\mu)|_{\Delta\mu} = -2n^2 + 2(2\lambda + 1)n - \lambda(2\lambda + 1) \tag{A.7}$$

holds. From (A.3) and (A.7), we obtain

$$|H_2(\mu)| - |D(\mu)\backslash H_2(\mu)| = -2n^2 + (4\lambda + 1)n - \lambda(2\lambda + 1). \tag{A.8}$$

Next, we apply the result to a partition having two or more columns with the same length. When a partition have m columns with the same length, we extract the $2[m/2]$ neighbouring columns from the original partition, where $[n]$ means the maximum integer that does not exceed n. Since the extracted $2[m/2]$ columns have the same number of the elements of $H_2(\mu)$ and $D(\mu)\backslash H_2(\mu)$, there is no contribution to $|H_2(\mu)| - |D(\mu)\backslash H_2(\mu)|$. We can use result (A.8) for the left partition with $2\lambda + 1 - 2[m/2]$ columns and obtain the equation by replacing λ in (A.8) by $\lambda - [m/2]$. The above discussion can be applied to a partition that has more than one set of columns with the same length. These applications, however, do not change the result on the condition $|C_2(\mu)| + |H_2(\mu)| = |\mu|$.

Appendix B. Energy spectrum and matrix elements

In this appendix, we derive (92) and (93) in appendix B.1 and (94), (94) and (96) in appendix B.2.

B.1. Energy spectrum

In the excitation energy ω_κ in (63), the energy

$$E_{N-1}(\kappa) = (2\pi/L)^2 \sum_{i=1}^{N-1} (\kappa_i + \lambda(N - 2\mathrm{i})/2)^2$$

of $N - 1$ particle state κ is rewritten as

$$E_{N-1}(\kappa) = (\pi/L)^2 \sum_{i=1}^{N-1} (-\mu_i + \lambda + (1 + 2\lambda)\,(i - N/2) + \alpha_{N-i} - 1)^2\,, \qquad \text{(B.1)}$$

using

$$\nu_i = \alpha_{N-i} - 2\kappa_{N-i} - N + 1 + i, \qquad \text{(B.2)}$$

which is (37) with replacement of N by $N - 1$ and

$$\mu = \nu + N/2 + \lambda - 2 \qquad \text{(B.3)}$$

coming from (71). In (B.1), the value of α_{N-i} is given by

$$\alpha_{N-i} = \begin{cases} 2, & (i, \mu_i) \in C_2(\mu) \\ 1, & (i, \mu_i) \in D(\mu)/C_2(\mu), \end{cases} \qquad \text{(B.4)}$$

which follows from

$$\begin{aligned} \alpha_{N-i} &= \mu_i - i + \underbrace{2\kappa_{N-i}}_{\text{even}} + \underbrace{N/2 + 1 - \lambda}_{\text{even}} \\ &\equiv a'(i, \mu_i) + l'(i, \mu_i) \quad \mod 2. \end{aligned} \qquad \text{(B.5)}$$

For $i \in [\mu_1' + 1, N - 1]$, we regard $(i, 0)$ as an element of $C_2(\mu)$ $(D(\mu)\backslash C_2(\mu))$ when i is even (odd). Using (B.4), $E_{N-1}(\kappa)$ is rewritten as

$$E_{N-1}(\kappa) = \frac{\pi^2(1 + 2\lambda)^2}{L^2}(\mathcal{E}'_{N-1}(\kappa) + \mathcal{E}''_{N-1}(\kappa)), \qquad \text{(B.6)}$$

with

$$\mathcal{E}'_{N-1}(\kappa) = \sum_{i \in [1,N-1];\text{s.t. } (i,\mu_i) \in C_2(\mu)} \left(\mathrm{i} - \frac{N}{2} + \frac{\lambda - \mu_i + 1}{2\lambda + 1}\right)^2 \qquad \text{(B.7)}$$

and

$$\mathcal{E}''_{N-1}(\kappa) = \sum_{i \in [1,N-1];\text{s.t. } (i,\mu_i) \in D(\mu)\backslash C_2(\mu)} \left(\mathrm{i} - \frac{N}{2} + \frac{\lambda - \mu_i}{2\lambda + 1}\right)^2. \qquad \text{(B.8)}$$

Now we consider $\mathcal{E}'_{N-1}(\kappa)$. It is convenient to decompose the sum with respect to i as

$$\sum_{\substack{i \in [1,N-1],\\ s.t. (i,\mu_i) \in C_2(\mu)}} \to \sum_{j=0}^{2\lambda+1} \sum_{\substack{i \in [1+\mu'_{j+1},\mu'_j],\\ s.t. (i,\mu_i) \in C_2(\mu)}} \qquad \text{(B.9)}$$

In the interval $i \in [1+\mu'_{j+1}, \mu'_j]$, i's satisfying $(i, \mu_i) \in C_2(\mu)$ align alternately. The minimum (maximum) value $i^{\square}_{\min}(j)$ ($i^{\square}_{\max}(j)$) of i in $[1+\mu_{j+1}, \mu_j]$ satisfying $(i, \mu_i) \in C_2(\mu)$ are given, respectively, by

$$i^{\square}_{\min}(j) = \mu'_{j+1} + \sigma_{j+1} + 3/2, \qquad i^{\square}_{\max}(j) = \mu'_j - \sigma_j - 1/2, \tag{B.10}$$

as shown below. First we note that $(i, \mu_i) = (i, j)$ in the interval $i \in [1+\mu'_{j+1}, \mu'_j]$. When $j+1 \in Q$, $\mu'_{j+1} - (j+1)$ is even and hence $(1+\mu'_{j+1}, j) \in C_2(\mu)$ and $i^{\square}_{\min}(j) = 1+\mu'_{j+1}$. When $j+1 \in P$, on the other hand, $\mu'_{j+1} - (j+1)$ is odd and hence $(1+\mu'_{j+1}, j) \in D(\mu)\backslash C_2(\mu)$ and $i^{\square}_{\min}(j) = 2+\mu'_{j+1}$. Thus we arrive at the first equation of (B.10). We can obtain the second equation of (B.10) in a similar way. In terms of $i^{\square}_{\min}(j)$ and $i^{\square}_{\max}(j)$, $\mathcal{E}'_{N-1}(\kappa)$ is written as

$$\begin{aligned}\mathcal{E}'_{N-1}(\kappa) &= \sum_{j=0}^{2\lambda+1} \sum_{i=i^{\square}_{\min}(j), i^{\square}_{\min}(j)+2,\ldots}^{i^{\square}_{\max}(j)} \left(\mathrm{i} - \frac{N}{2} + \frac{\lambda-j+1}{2\lambda+1}\right)^2 \\ &= \frac{1}{6}\sum_{j=0}^{2\lambda+1}\left(i^{\square}_{\max}(j) + 1 - N/2 + \frac{\lambda-j+1}{2\lambda+1}\right)^3 \\ &\quad - \frac{1}{6}\sum_{j=0}^{2\lambda+1}\left(i^{\square}_{\min}(j) - 1 - N/2 + \frac{\lambda-j+1}{2\lambda+1}\right)^3 \\ &\quad + \frac{1}{6}\sum_{j=0}^{2\lambda+1}\left(-i^{\square}_{\max}(j) + i^{\square}_{\min}(j) - 2\right).\end{aligned} \tag{B.11}$$

In the second equality, we have used the following formula:

$$\sum_{i=m,m+2,m+4,\ldots,n} (i+A)^2 = \frac{1}{6}[(n+1+A)^3 - (m-1+A)^3 - n + m - 2] \tag{B.12}$$

for n, m satisfying $n - m$ being a positive and even integer. Expression (B.11) can be further rewritten as

$$\begin{aligned}\mathcal{E}'_{N-1}(\kappa) &= \frac{1}{6}\sum_{j=1}^{2\lambda+1}[(\tilde{\mu}'_j - \sigma_j)^3 - (\tilde{\mu}'_j + \sigma_j + 1/(2\lambda+1))^3 + 2\sigma_j] \\ &\quad + \frac{M(M^2-1)}{3} + \frac{\lambda^2 M}{(2\lambda+1)^2} + \frac{1}{6}\end{aligned} \tag{B.13}$$

using (B.10) and (91). We can rewrite expression (B.8) for $\mathcal{E}''_{N-1}(\kappa)$ in a similar way. First (B.8) is rewritten as

$$\begin{aligned}\mathcal{E}''_{N-1}(\kappa) &= \sum_{j=0}^{2\lambda+1} \sum_{\substack{i\in[1+\mu'_{j+1}, \mu'_j],\\ s.t.(i,\mu_i)\in D(\mu)\backslash C_2(\mu)}} \left(\mathrm{i} - \frac{N}{2} + \frac{\lambda-j}{2\lambda+1}\right)^2 \\ &= \sum_{j=0}^{2\lambda+1} \sum_{i=i^{\blacksquare}_{\min}(j), i^{\blacksquare}_{\min}(j)+2,\ldots}^{i^{\blacksquare}_{\max}(j)} \left(\mathrm{i} - \frac{N}{2} + \frac{\lambda-j}{2\lambda+1}\right)^2, \\ &= \frac{1}{6}\sum_{j=0}^{2\lambda+1}\left(i^{\blacksquare}_{\max}(j) + 1 - N/2 + \frac{\lambda-j}{2\lambda+1}\right)^3\end{aligned}$$

$$- \frac{1}{6} \sum_{j=0}^{2\lambda+1} \left(i^{\blacksquare}_{\min}(j) - 1 - N/2 + \frac{\lambda - j}{2\lambda + 1} \right)^3$$
$$+ \frac{1}{6} \sum_{j=0}^{2\lambda+1} \left(-i^{\blacksquare}_{\max}(j) + i^{\blacksquare}_{\min}(j) - 2 \right), \tag{B.14}$$

where

$$i^{\blacksquare}_{\min}(j) = \mu'_{j+1} - \sigma_{j+1} + 3/2, \qquad i^{\blacksquare}_{\max}(j) = \mu'_j + \sigma_j - 1/2 \tag{B.15}$$

are, respectively, defined as the minimum and maximum of i satisfying $(i, \mu_i) \in D(\mu)/C_2(\mu)$ in $[1 + \mu'_{j+1}, \mu'_j]$. In the last equality in (B.14), we have used (B.12). Substituting (B.15) into (B.14) and using (91), $\mathcal{E}''_{N-1}(\kappa)$ is expressed as

$$\mathcal{E}''_{N-1}(\kappa) = \frac{1}{6} \sum_{j=1}^{2\lambda+1} [(\tilde{\mu}'_j + \sigma_j - 1/(2\lambda + 1))^3 - (\tilde{\mu}'_j - \sigma_j)^3 - 2\sigma_j]$$
$$+ \frac{M(M^2 - 1)}{3} + \frac{\lambda^2 M}{(2\lambda + 1)^2} + \frac{1}{6}. \tag{B.16}$$

Substituting (B.13) and (B.16) into (B.6), we obtain

$$E_{N-1}(\kappa) = -(2\lambda + 1) \sum_{j=1}^{2\lambda+1} \left(\frac{\pi(\tilde{\mu}'_j + \sigma_j)}{L} \right)^2 + \frac{4\pi^2 \lambda(\lambda + 1)}{3L^2}$$
$$+ \underbrace{\frac{2\pi^2 (2\lambda + 1)^2 M(M^2 - 1)}{3L^2} + \frac{2\pi^2 \lambda^2 M}{L^2}}_{= E_N(\mathrm{g})} \tag{B.17}$$

from which and (63), expressions (92) and (93) follow.

B.2. Matrix elements

Here, we describe $X_\mu, Y_\mu(r), Z_\mu(r)$ in the matrix elements in terms of the renormalized momenta (91) and the spin variables (89) and (90). First we consider X_μ. The product with respect to $s = (i, j) \in D(\mu)\backslash C_2(\mu)$ is taken within each column, and then taken over the column. In the jth column, this condition is equivalent to even (odd) i when j is odd (even). The square $s = (\mu'_j, j)$ belongs to $D(\mu)\backslash C_2(\mu)$ when $j \in P$, and $s = (\mu'_j - 1, j)$ belongs to $D(\mu)\backslash C_2(\mu)$ when $j \in Q$. The maximum value of i in the jth column is thus expressed as

$$i_{\max} = \mu'_j - \delta_{\sigma_j \sigma_{2\lambda+2}}. \tag{B.18}$$

The contribution to X_μ from the jth column is then given by

$$\prod_{i:\mathrm{even}}^{i_{\max}} (-\alpha(j-1) + i/2) = \prod_{i'=1}^{i_{\max}/2} (i' - \alpha(j-1))$$
$$= \frac{\Gamma[i_{\max}/2 + 1 - \alpha(j-1)]}{\Gamma[1 - \alpha(j-1)]} \tag{B.19}$$

for odd j and

$$\prod_{i:\mathrm{odd}}^{i_{\max}} (-\alpha(j-1) + i/2) = \prod_{i'=1}^{(i_{\max}+1)/2} (i' - 1/2 - \alpha(j-1))$$
$$= \frac{\Gamma[i_{\max}/2 + 1 - \alpha(j-1)]}{\Gamma[1/2 - \alpha(j-1)]} \tag{B.20}$$

for even j. We introduce a dummy index $i' = i/2$ in (B.19) and $i' = (i+1)/2$ in (B.20), respectively. From (B.19) and (B.20) and using

$$\prod_{j\in[1,3,\ldots,2\lambda+1]} \Gamma[1-\alpha(j-1)] \prod_{j\in[2,4,\ldots,2\lambda]} \Gamma[1/2-\alpha(j-1)] = \prod_{j=1}^{2\lambda+1} \Gamma[j/(2\lambda+1)], \tag{B.21}$$

we obtain

$$X_\mu = \prod_{j=1}^{2\lambda+1} \frac{\Gamma[i_{\max}/2+1-\alpha(j-1)]}{\Gamma[j/(2\lambda+1)]}. \tag{B.22}$$

With the use of (B.18) and (91), (B.22) becomes (94).

Similarly, we can write $Y_\mu(r)$ in terms of $\tilde{\mu}'_j$ and σ_j for $j \in [0, 2\lambda+1]$, as shown below. When $(i, j) \in C_2(\mu)$, both i and j are odd or even. The maximum value $i_{\max}$ of i in the jth column is given by $i_{\max} = \mu'_j - \delta_{\sigma_j \sigma_0}$. The contribution to $Y_\mu(r)$ from the jth column is given by

$$\begin{aligned}\prod_{i:\mathrm{odd}}^{i_{\max}} (\alpha(j-1)+r+(N-i)/2) &= \prod_{i'=1}^{(i_{\max}+1)/2} (\alpha(j-1)+r+(N+1)/2-i') \\ &= \frac{\Gamma[\alpha(j-1)+r+(N+1)/2]}{\Gamma[\alpha(j-1)+r+(N-i_{\max})/2]}\end{aligned} \tag{B.23}$$

for odd j and

$$\begin{aligned}\prod_{i:\mathrm{even}}^{i_{\max}} (\alpha(j-1)+r+(N-i)/2) &= \prod_{i'=1}^{i_{\max}/2} (\alpha(j-1)+r+N/2-i') \\ &= \frac{\Gamma[\alpha(j-1)+r+N/2]}{\Gamma[\alpha(j-1)+r+(N-i_{\max})/2]}\end{aligned} \tag{B.24}$$

for even j. With the use of (91) and the relation

$$\begin{aligned}&\prod_{j\in[1,3,\ldots,2\lambda+1]} \Gamma[\alpha(j-1)+r+(N+1)/2] \prod_{j\in[2,4,\ldots,2\lambda]} \Gamma[\alpha(j-1)+r+N/2] \\ &\quad = \prod_{j=1}^{2\lambda+1} \Gamma[\alpha+r+N/2+(j-1)/(2\lambda+1)],\end{aligned} \tag{B.25}$$

the expression $Y_\mu(r)$ is obtained as (95).

Next we consider the expression for $Z_\mu(r)$ in terms of $\{\tilde{\mu}'_j, \sigma_j\}$. It is convenient to decompose $D(\mu)$ as shown in figure B1. Correspondingly, $Z_\mu(r)$ is rewritten as

$$Z_\mu(r) = \prod_{j=1}^{2\lambda+1} \prod_{k=j}^{2\lambda+1} \prod_{s\in H_2(\mu)\cap D_{jk}} (\alpha a(s)+r+l(s)/2) \tag{B.26}$$

with

$$D_{jk} = \{s = (i, j) | i \in [1+\mu'_{k+1}, \mu'_k]\}. \tag{B.27}$$

Within D_{jk}, the arm length $a(s)$ is constant ($=k-j$) and hence the squares which belong to $H_2(\mu)$ and $D(\mu)\backslash H_2(\mu)$ are, respectively, aligned alternately. Let $i_{\min}$ and $i_{\max}$ be, respectively, the minimum and maximum values of i in $s \in H_2(\mu) \cap D_{jk}$. Thus, the product

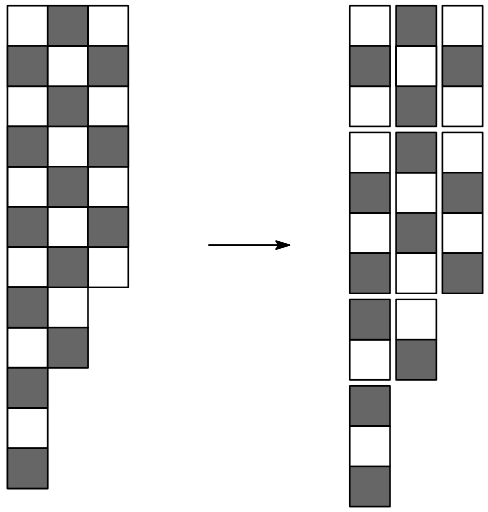

Figure B1. The Young diagram of a quasi-hole state for $\lambda = 1$ is decomposed into D_{jk} for calculation of $Z_\mu(r)$.

over $s = (i, j) \in H_2(\mu) \cap D_{jk}$ is expressed as the product over $i' = 0, 1, \ldots, (i_{\max} - i_{\min})/2$, with $i = 2i' + i_{\min}$. The contribution from D_{jk} to $Z_\mu(r)$ is given by

$$\prod_{s\in H_2(\mu)\cap D_{jk}} (\alpha a(s) + r + l(s)/2) = \sum_{i'=0}^{(i_{\max}-i_{\min})/2} (\alpha(k-j) + r + \mu'_j/2 - i_{\min}/2 - i')$$
$$= \frac{\Gamma[\alpha(k-j) + r + \mu'_j/2 - i_{\min}/2 + 1]}{\Gamma[\alpha(k-j) + r + \mu'_j/2 - i_{\max}/2]}. \tag{B.28}$$

The maximum value $i_{\max}$ of i in $H_2(\mu) \cap D_{jk}$ is expressed as

$$i_{\max} = \mu'_k - \delta_{\sigma_j\sigma_k} \tag{B.29}$$

as shown below.

For $s \in D_{jk}$, $a(s)$ is $k - j$ and $l(s) = \mu'_j - i$, and hence $h(s) = a(s) + l(s) + 1$ is written as

$$h(s) = \mu'_j - j + k - i + 1 = \mu'_j - j - (\mu'_k - k) + \mu'_k - i + 1. \tag{B.30}$$

When $(j, k) \in (P, P)$ or (Q, Q), both $\mu'_j - j$ and $\mu'_k - k$ are odd or even and hence for $(i, j) \in H_2(\mu) \cap D_{jk}$,

$$\mu'_k - 1 \equiv i \qquad \text{mod } 2. \tag{B.31}$$

When $(j, k) \in (P, Q)$ or (Q, P), on the other hand, the sum of $\mu'_j - j$ and $\mu'_k - k$ is odd and hence,

$$\mu'_k \equiv i \qquad \text{mod } 2. \tag{B.32}$$

The two relations (B.31) and (B.32) are expressed as (B.29) in a unified way. In a similar way, $i_{\min}$ is expressed as

$$i_{\min} = \mu'_{k+1} + 1 + \delta_{\sigma_j\sigma_{k+1}} \tag{B.33}$$

respectively.

Expression (B.28) is further written as

$$\prod_{s\in H_2(\mu)\cap D_{jk}} (\alpha a(s) + r + l(s)/2) = \frac{\Gamma\left[\left(\tilde{\mu}'_j - \tilde{\mu}'_{k+1} - \delta_{\sigma_j\sigma_{k+1}}\right)/2 + r - \alpha + 1/2\right]}{\Gamma\left[\left(\tilde{\mu}'_j - \tilde{\mu}'_k + \delta_{\sigma_j\sigma_k}\right)/2 + r\right]}, \quad \text{(B.34)}$$

with the use of (B.29), (B.33) and (91). From (B.34), expression (96) for $Z_\mu(r)$ follows.

Appendix C. Spectral weight

The triple integral in the spectral function for $\lambda = 1$,

$$A^-(\epsilon, p) = C\int_{-1}^{1} du_1 \int_{-1}^{1} du_2 \int_{-1}^{1} du_3\, \delta\left(p - \frac{\pi d}{2}(u_1+u_2+u_3)\right) \times \delta\left(\epsilon - \frac{3(\pi d)^2}{4}\left(3 - u_1^2 - u_2^2 - u_3^2\right)\right) F(u_1, u_2, u_3) \quad \text{(C.1)}$$

with

$$F(u_1, u_2, u_3) = |u_2 - u_3|^{4/3}|u_1 - u_3|^{-2/3}|u_1 - u_2|^{-2/3}\prod_{j=1}^{3}\left(1 - u_j^2\right)^{-1/3} \quad \text{(C.2)}$$

reduces to an integral on a curve determined by the sphere, plane, and the cube

$$\epsilon = \frac{3(\pi d)^2}{4}\left(3 - u_1^2 - u_2^2 - u_3^2\right) \quad \text{(C.3)}$$

$$p = \frac{\pi d}{2}(u_1 + u_2 + u_3) \quad \text{(C.4)}$$

$$|u_i| \leqslant 1, \qquad i = 1, 2, 3. \quad \text{(C.5)}$$

The cross section between (C.4) and (C.5) is triangular when

$$\pi d/2 \leqslant p \leqslant 3\pi d/2 \quad \text{(C.6)}$$

and hexagon when

$$0 \leqslant p \leqslant \pi d/2, \quad \text{(C.7)}$$

as shown in figure C1.

In the following, we consider the case (C.6) only. Case (C.7) can be discussed similarly. The integral in (C.1) is on a circle when

$$\frac{\pi d}{2} \leqslant p \leqslant \frac{3\pi d}{2}, \qquad \text{and} \qquad -\frac{3}{2}\left(p - \frac{\pi d}{2}\right)^2 + \frac{3}{2}(\pi d)^2 \leqslant \epsilon \leqslant -p^2 + \frac{9}{4}(\pi d)^2, \quad \text{(C.8)}$$

and it is on three disconnected pieces of arc when

$$\pi d/2 \leqslant p \leqslant 3\pi d/2$$

and

$$-3(p - \pi d)^2 + \frac{3}{4}(\pi d)^2 \leqslant \epsilon \leqslant -\frac{3}{2}\left(p - \frac{\pi d}{2}\right)^2 + \frac{3}{2}(\pi d)^2,$$

as shown by bold curves in figure C2. The integrand $F(u_1, u_2, u_3)$ in (C.1) diverges when

$$u_1 = u_3, \qquad u_1 = u_2 \qquad \text{or} \qquad |u_j| = 1 \quad \text{for} \quad j = 1, 2, 3, \quad \text{(C.9)}$$

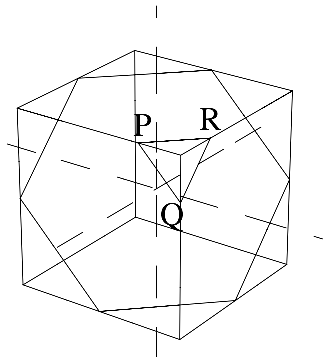

Figure C1. Cross section of the plane (C.4) and the cube (C.5). The cross section is triangular PQR when $p \in [\pi d/2, 3\pi d/2]$ and hexagon when $p \in [0, \pi d/2]$.

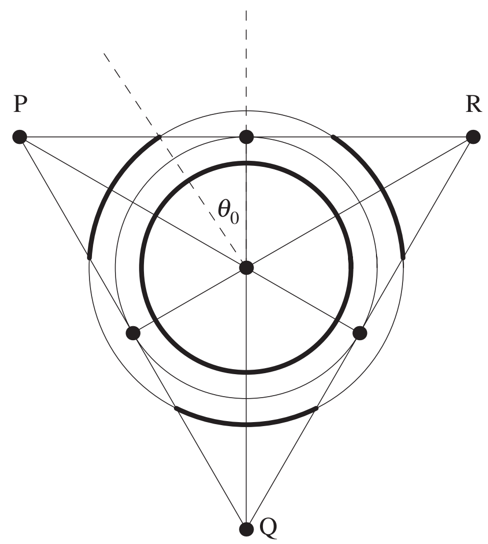

Figure C2. Cross section of the plane (C.4) and the cube (C.5) for $p \in [\pi d/2, 3\pi d/2]$.

which are represented by solid lines in figure C2. When the contour of the integral passes near the crossing points of the above lines (C.9), the spectral function becomes singular. Those crossing points are given by

$$(u_1, u_2, u_3) = \left(\frac{2p}{3\pi d}, \frac{2p}{3\pi d}, \frac{2p}{3\pi d}\right), \tag{C.10}$$

$$\left(1, 1, -2 + \frac{2p}{\pi d}\right), \tag{C.11}$$

$$\left(-\frac{1}{2} + \frac{p}{\pi d}, -\frac{1}{2} + \frac{p}{\pi d}, 1\right) \tag{C.12}$$

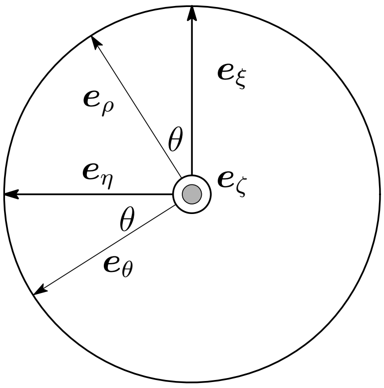

Figure C3. Cartesian coordinate $\{e_\xi, e_\eta, e_\zeta\}$ and cylindrical coordinate $\{e_\rho, e_\theta, e_\zeta\}$.

and their equivalent points. They are depicted in figure C1 by dots. The singularity of the spectral function near the upper edge $\epsilon = (3\pi d/2)^2(1-(2p/(\pi d))^2)$ of the support comes from (C.10). Point (C.11) yields the singularity of $A(\epsilon, p)$ near the lower edge $\epsilon = 3(\pi d/2)^2(2p/(\pi d)-1)(3-2p/(\pi d))$. (C.12) is relevant to the singularity near

$$\epsilon = \frac{3}{2}\left(p+\frac{\pi d}{2}\right)\left(\frac{3\pi d}{2}-p\right) \equiv \epsilon_p. \tag{C.13}$$

The singularities near the upper edge and lower edge have been obtained in earlier papers [46, 62]. We thus consider the singularity when $\delta\epsilon \equiv \epsilon - \epsilon_p \sim 0$ in the following.

Now we introduce the Cartesian coordinate $e_i \cdot e_j = \delta_{ij}$ with $i, j = 1, 2, 3$ and define the vector

$$u = u_1 e_1 + u_2 e_2 + u_3 e_3. \tag{C.14}$$

We define another Cartesian coordinate

$$\begin{aligned} e_\xi &= -e_1/\sqrt{6} - e_2/\sqrt{6} + 2e_3/\sqrt{6}, \\ e_\eta &= -e_1/\sqrt{2} + e_2/\sqrt{2}, \\ e_\zeta &= (e_1 + e_2 + e_3)/\sqrt{3} \end{aligned} \tag{C.15}$$

and circular coordinate on $e_\xi - e_\eta$ plane

$$e_\rho = e_\xi \cos\theta + e_\eta \sin\theta, \qquad e_\theta = -e_\xi \sin\theta + e_\eta \cos\theta. \tag{C.16}$$

Note that $\{e_\xi, e_\eta, e_\zeta\}$ forms a cylindrical coordinate as shown in figure C3. In terms of (C.15) and (C.16), we rewrite u as

$$u = u_\rho e_\rho + u_\zeta e_\zeta. \tag{C.17}$$

In the following, we change the variables of integral in (C.1) from (u_1, u_2, u_3) to $(u_\rho, \theta, u_\zeta)$. From (C.14)–(C.17), we obtain

$$\begin{aligned} u_1 &= u \cdot e_1 = u_\rho(-\cos\theta/\sqrt{6} - \sin\theta/\sqrt{2}) + u_\zeta/\sqrt{3} \\ u_2 &= u \cdot e_2 = u_\rho(-\cos\theta/\sqrt{6} + \sin\theta/\sqrt{2}) + u_\zeta/\sqrt{3} \\ u_3 &= u \cdot e_3 = 2u_\rho \cos\theta/\sqrt{6} + u_\zeta/\sqrt{3}. \end{aligned} \tag{C.18}$$

The spectral function (C.1) is rewritten as

$$A^-(\epsilon, p) = C\int \mathrm{d}u_\rho u_\rho \int \mathrm{d}\theta \int \mathrm{d}u_\zeta \delta\left(p - \frac{\sqrt{3}\pi d u_\zeta}{2}\right) \times \delta\left(\epsilon - 3(\pi d/2)^2\left(3 - u_\rho^2 - u_\zeta^2\right)\right) F(u_1, u_2, u_3). \tag{C.19}$$

From the delta functions, u_ρ and u_ζ are forced to be

$$u_\rho = \sqrt{3 - 4(\epsilon + p^2)/(3\pi^2 d^2)}, \qquad u_\zeta = 2p/(\sqrt{3}\pi d), \tag{C.20}$$

respectively. As a result, $A^-(\epsilon, p)$ becomes

$$A^-(\epsilon, p) = \frac{4C}{(\sqrt{3}\pi d)^3}\int \mathrm{d}\theta F(\bar{u}_1(\theta), \bar{u}_2(\theta), \bar{u}_3(\theta)) \tag{C.21}$$

with

$$\begin{aligned} \bar{u}_1(\theta) &= \sqrt{(1/2 - p'/3)^2 - 2\delta\epsilon'/9}(-\cos\theta - \sqrt{3}\sin\theta) + 2p'/3 \\ \bar{u}_2(\theta) &= \sqrt{(1/2 - p'/3)^2 - 2\delta\epsilon'/9}(-\cos\theta + \sqrt{3}\sin\theta) + 2p'/3 \\ \bar{u}_3(\theta) &= \sqrt{(1 - 2p'/3)^2 - 4\delta\epsilon'/9}\cos\theta + 2p'/3. \end{aligned} \tag{C.22}$$

Here, we have introduced $\delta\epsilon' \equiv \delta\epsilon/(\pi d)^2$ and $p' = p/(\pi d)$. The integral in (C.21) runs over

$$\begin{aligned} &\theta \in [0, 2\pi], \qquad \text{for} \quad \delta\epsilon > 0 \\ &\theta \in [\theta_0, 2\pi/3 - \theta_0] \cup [2\pi/3 + \theta_0, 4\pi/3 - \theta_0] \cup [4\pi/3 + \theta_0, 2\pi - \theta_0] \end{aligned} \tag{C.23}$$

for $\delta\epsilon < 0$ with

$$\theta_0 = \arcsin\left[\sqrt{\frac{-2\delta\epsilon'}{(3/2 - p')^2 - 2\delta\epsilon'}}\right].$$

Point (C.12) and equivalent points

$$\left(1, -\frac{1}{2} + \frac{p}{\pi d}, -\frac{1}{2} + \frac{p}{\pi d}\right), \qquad \left(-\frac{1}{2} + \frac{p}{\pi d}, 1, -\frac{1}{2} + \frac{p}{\pi d}\right) \tag{C.24}$$

correspond to $\delta\epsilon' = 0$ and $\theta = 0, 2\pi/3, 4\pi/3$, respectively. The most singular contribution to $A^-(\epsilon, p)$ for $\delta\epsilon = 0$ comes from the vicinity of $\theta = 0$, where the factors $|u_1 - u_2|^{-2/3}(1 - u_3)^{-1/3}$ become singular in $F(u_1, u_2, u_3)$. Therefore, we approximate $F(u_1, u_2, u_3)$ as

$$F \propto (\bar{u}_1(\theta) - \bar{u}_2(\theta))^{-2/3}(1 - \bar{u}_3(\theta))^{-1/3} \propto \frac{|\sin\theta|^{-2/3}}{(1 - g\cos\theta)^{1/3}}, \tag{C.25}$$

with $g = \sqrt{1 - \delta\epsilon'/(p' - 3/2)^2}$. $A^-(\epsilon, p)$ is evaluated as

$$A^-(\epsilon, p) \propto \int_0^{\theta_c} \frac{|\sin\theta|^{-2/3}}{(1 - g\cos\theta)^{1/3}}\,\mathrm{d}\theta \tag{C.26}$$

for $\delta\epsilon > 0$ and $\delta\epsilon \sim 0$ and

$$A^-(\epsilon, p) \propto \int_{\theta_0}^{\theta_c} \frac{|\sin\theta|^{-2/3}}{(1 - g\cos\theta)^{1/3}}\,\mathrm{d}\theta \tag{C.27}$$

for $\delta\epsilon < 0$ and $\delta\epsilon \sim 0$. Here, θ_c is a cut-off angle of the order unity. We can take $\theta_c = \pi/3$, or $\pi/2$, for example. First we evaluate (C.26). Introducing $t = \tan(\theta/2)$, $t_c = \tan(\theta_c/2)$ and $\tilde{g} = (1 - g)/(1 + g)$, (C.26) becomes

$$A^-(\epsilon, p) \propto \int_0^{t_c} \frac{t^{-2/3}(1 + t^2)^{-4/3}}{(t^2 + \tilde{g})^{1/3}}\,\mathrm{d}t. \tag{C.28}$$

We rewrite (C.28) as

$$\int_0^{t_c} \frac{t^{-2/3}(1+t^2)^{-4/3}}{(t^2+\tilde{g})^{1/3}}\,\mathrm{d}t \tag{C.29}$$

$$= \int_0^{t_c} \frac{t^{-2/3}}{(t^2+\tilde{g})^{1/3}}\,\mathrm{d}t + \int_0^{t_c} \frac{t^{-2/3}((1+t^2)^{-4/3}-1)}{(t^2+\tilde{g})^{1/3}}\,\mathrm{d}t \tag{C.30}$$

The second integral of the right-hand side converges when $\tilde{g} = 0$. The first term of the right-hand side, on the other hand, is rewritten as

$$\tilde{g}^{-1/6}\int_0^{t_c\tilde{g}^{-1/2}} \frac{\tau^{-2/3}}{(\tau^2+1)^{1/3}}\,\mathrm{d}\tau \sim \tilde{g}^{-1/6}\int_0^{\infty} \frac{\tau^{-2/3}}{(\tau^2+1)^{1/3}}\,\mathrm{d}\tau. \tag{C.31}$$

Since $\tilde{g} \propto \delta\epsilon$, we arrive at

$$A^{-}(\epsilon, p) \propto (\delta\epsilon)^{-1/6} \tag{C.32}$$

when $\delta\epsilon > 0$ and $\delta\epsilon \sim 0$.

The singularity near $\delta\epsilon = 0$ and $\delta\epsilon < 0$ is evaluated similarly. Introducing $t_0 = \tan(\theta_0/2)$, (C.27) becomes

$$A^{-}(\epsilon, p) \propto \int_{t_0}^{t_c} \frac{t^{-2/3}(1+t^2)^{-4/3}}{(t^2+\tilde{g})^{1/3}}\,\mathrm{d}t \sim \tilde{g}^{-1/6}\int_{t_0\tilde{g}^{-1/2}}^{t_c\tilde{g}^{-1/2}} \frac{\tau^{-2/3}}{(\tau^2+1)^{1/3}}\,\mathrm{d}\tau \tag{C.33}$$

When $\tilde{g} \sim 0$, equivalently $\delta\epsilon \sim 0$, $t_0\tilde{g}^{-1/2}$ is the order of unity and hence,

$$A^{-}(\epsilon, p) \propto (-\delta\epsilon)^{-1/6} \tag{C.34}$$

when $\delta\epsilon < 0$ and $\delta\epsilon \sim 0$.

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